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GEOMETRY
OF
TIME AND SPACE

LONDON
Cambridge University Press
FETTER LANE

NEW YORK · TORONTO
BOMBAY · CALCUTTA · MADRAS
Macmillan

TOKYO
Maruzen Company Ltd

GEOMETRY
OF
TIME AND SPACE

by

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CAMBRIDGE
AT THE UNIVERSITY PRESS
1936

"The Bird of Time has but a little way
To fly—and Lo! the Bird is on the Wing."
OMAR KHAYYÁM

"I could not have been in two places at once unless I were a bird." SIR BOYLE ROCHE

Contrary to the view so generally held; not even "the Bird of Time" can be in two places at once. AUTHOR

PRINTED IN GREAT BRITAIN

PREFACE

THE present volume is essentially a second edition of one which was published by the author in 1914 under the title: *A Theory of Time and Space*. An alteration of the title has been made, since it was considered that the word *geometry* conveyed a somewhat better idea of the nature of the contents of the book than did the word *theory*.

The first edition was going through the press at the time of the outbreak of the war, so that its publication took place under very unfavourable circumstances. The present volume differs from its predecessor in several respects. The Introduction has been re-written and extended; while the proofs of a number of theorems, which were rather lengthy, have been curtailed and simplified.

A considerable amount of new matter has also been introduced, making the book more self-contained and complete.

The demonstrations have all been carried out as deductions from certain postulates expressed in terms of the relations of *after* and *before*; so that the whole work may be regarded as a demonstration of the fundamental character of these relations in Time-Space theory.

So far as I am aware, the book, in its original form, was the first of its kind to be written, and a brief account of its origin may be of interest. At the meeting of the British Association held at Belfast in 1902, Lord Rayleigh gave a paper entitled: *Does Motion through the Ether cause double Refraction?* in which he described certain experiments which he had carried out with the object of testing this matter, and which seemed to indicate that the answer was in the negative.

I remember that he inquired of Professor Larmor, who was present on this occasion, whether, from his theory, he would expect double refraction to be produced in this way. Professor Larmor replied that he would not, and, in the discussion which followed considerable surprise was expressed that, in any attempt to detect motion through the aether, things seemed to conspire together so as to give null results. The impression which this discussion made upon me was, that, in order properly to understand the matter, it would be necessary to make some sort of analysis of one's ideas concerning equality of

lengths, etc., and I decided that, at some future time, I should attempt to carry this out. I am not quite certain that I had not some idea of the sort prior to this meeting, but, in any case, the inspiration came from Professor Larmor, either then, or on some previous occasion while attending his lectures.

Some years later I attempted to carry out this scheme, and, while doing so, I heard for the first time of Einstein's work.

I may say that, from the first, I felt dissatisfied with his approach to the subject, and I decided to continue my own efforts to find a suitable basis for a theory.

The first work which I published on the subject was a pamphlet which appeared in 1911 entitled: *Optical Geometry of Motion: A New View of the Theory of Relativity.*

This pamphlet was of an exploratory character and did not profess to give a complete logical analysis of the subject; but nevertheless, although bearing a very different aspect, it contained some of the germs of my later work. It was, in fact, an attempt to describe Time-Space relations without making any assumption as to the simultaneity of events at different places. Later on, the idea of *Conical Order* occurred to me, in which instants at different places are regarded as definitely distinct; so that there is no such simultaneity.

As it was evident that a thorough working out of this idea would entail a great deal of labour, I published, in 1913, a short preliminary account of it under the title: *A Theory of Time and Space.*

In 1914, as above mentioned, I published a book bearing the same title, of which the present volume is a second edition.

The working out of a scientific theory in the form of a sequence of propositions, such as was done by Euclid, Newton and others, seems largely to have gone out of vogue in these latter days and I consider that this is rather regrettable.

No doubt, in doing exploratory work, other methods are permissible and necessary, but I think that the incorporation of the more fundamental parts of a theory in a sequence of propositions should always be kept in view, since, in this way, one is able to see much more readily what are our primary assumptions, and one is able to fall back upon these in cases of difficulty.

One can also test the effect on a theory of an alteration in one or

more of these primary assumptions such, for instance, as that produced in ordinary geometry by the rejection of the Euclidean axiom of parallels and the substitution for it of some other primary assumption, such as that of Lobatschewski. It will be found that the theory developed in this work is dependent upon the rejection of one generally accepted postulate with regard to instants of time and the substitution of others.

In conclusion, I desire once more to express my indebtedness to Sir Joseph Larmor, without whom this book would never have been written; and to convey my best thanks to the officials and staff of the Cambridge University Press for the care and skill with which they have carried out the printing.

ALFRED A. ROBB

CAMBRIDGE
20 *November* 1935

INTRODUCTION

In beginning the study of Geometrical Science it is customary to start with a course of pure geometry and, when a foundation of this has been laid, to proceed to the introduction of coordinate methods.

Thus, before being introduced to Cartesian geometry, one is taught certain propositions concerning the congruence of triangles, the properties of parallels, the theorem of Pythagoras, the theory of proportion, etc. To a large extent the methods of pure and of coordinate geometry are then carried on side by side, and it is customary, in proving a proposition, to make use of whichever method appears to be more convenient for the particular purpose in hand.

Speaking generally, no doubt, this is the course of procedure by which progress is most rapidly made, but I do not think that anyone would have the temerity to suggest that coordinate methods should be taken up without some prior grounding in pure geometry.

When one goes on to the study of other types of geometry than the Euclidean, the importance of logical sequence should become apparent, but I am sorry to say that it does not always seem to do so.

In many discussions of Time-Space theory we find ideas of ordinary Euclidean geometry carried forward into a domain in which they no longer apply, with occasional disastrous results.

The extension of Cartesian coordinates from three to four or more dimensions does not offer any very serious difficulties, since the formula for the square of the distance between two points, which, in three dimensions, has the form

$$s^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$$

becomes simply

$$s^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (w_1 - w_2)^2$$

in four dimensions; with a similar extension for any larger number.

It was found, however, by Minkowski that many of the facts connected with Time-Space theory could conveniently be represented by a four-dimensional coordinate geometry in which a formula

$$s^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - (t_1 - t_2)^2$$

held; that is to say, a formula in which one square is affected with the negative sign.

This negative sign makes an enormous difference in the subject and renders invalid a great part of what holds in ordinary Euclidean geometry.

Some idea of the extent of the modifications required may perhaps be obtained when I state that the construction of the very first proposition of Euclid becomes impossible except in a certain type of plane, and that two other types of plane occur in which an equilateral triangle cannot exist.

Numerous other features of this Time-Space geometry are so curious as to seem at first quite paradoxical, and some consideration of a few of these from the coordinate standpoint may perhaps emphasise the importance of laying a proper foundation for the subject on the purely geometrical side.

It is to be observed in the first place that whereas the expression

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2$$

(which may briefly be written in the form

$$\delta x^2+\delta y^2+\delta z^2)$$

is always positive for real values of δx, δy, δz which are not all zero; the expression

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2-(t_1-t_2)^2$$

(or

$$\delta x^2+\delta y^2+\delta z^2-\delta t^2)$$

may be either positive, zero or negative for real values of δx, δy, δz, δt differing from zero.

Three types of line joining the points (x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2) exist corresponding to these three cases.

When the expression is positive the square of the distance between the points is given by the formula

$$\delta s^2=\delta x^2+\delta y^2+\delta z^2-\delta t^2.$$

When the expression is negative, then, analytically, δs becomes a pure imaginary; but if we write

$$\delta \bar{s}^2=-\delta s^2$$

and recollect that we are now dealing with a line of a different type, we get the square of the distance in these new units given by

$$\delta \bar{s}^2=\delta t^2-\delta x^2-\delta y^2-\delta z^2.$$

When the expression is zero one is tempted to think that the distance

between the points must be zero; but this is a misleading interpretation. The real interpretation of the equation

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2-(t_1-t_2)^2=0$$

is that the points (x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2) lie in a particular type of line.

For this type of line the conception of length partially, but not entirely, breaks down.

We may compare lengths along a given line of this kind, or along two such parallel lines, but not along two which are not parallel.

Consider the case of lines for which the expression

$$\delta x^2+\delta y^2+\delta z^2-\delta t^2$$

is positive.

It is obvious that the axes of x, y and z (but *not* the axis of t), are lines of this character.

Now let O be the origin of coordinates and let P be any point in the positive axis of x and let $OP=2l$.

Let A_1, A_2 and A_3 be three points whose coordinates are given by the following table:

	A_1	A_2	A_3
x	l	l	l
y	b	c	0
z	0	0	0
t	0	c	Kl

(where $K^2<1$).

Then
$$OA_1{}^2=l^2+b^2 \qquad \therefore\ OA_1>l,$$
$$OA_2{}^2=l^2 \qquad \therefore\ OA_2=l,$$
$$OA_3{}^2=(1-K^2)\,l^2 \qquad \therefore\ OA_3<l.$$

Similarly
$$PA_1>l,$$
$$PA_2=l,$$
$$PA_3<l.$$

Thus
$$OA_1+PA_1>OP,$$
$$OA_2+PA_2=OP,$$
$$OA_3+PA_3<OP,$$

so that, in this geometry, we have two sides of a triangle together greater than, equal to, or less than the third. But the side OP is common

to all three triangles, so that its length is neither a minimum nor a maximum.

The question naturally arises: If such a line be neither a minimum nor a maximum, what is it? Our ordinary idea of a straight line breaks down.

The case is rather different if we take a triangle one of whose sides is a part of the axis of t. Thus let Q be any point on the positive axis of t such that $OQ = \lambda$ and let A be a point whose coordinates are (a, b, c, d), where $\lambda > d > 0$.

In order that the three sides of our triangle may be all lines of the same kind we shall suppose that a, b and c are so small that

$$a^2 + b^2 + c^2 < d^2,$$

and also
$$a^2 + b^2 + c^2 < (\lambda - d)^2.$$

Then
$$OA = \sqrt{d^2 - a^2 - b^2 - c^2} < d,$$

and
$$AQ = \sqrt{(\lambda - d)^2 - a^2 - b^2 - c^2} < \lambda - d.$$

Thus
$$OA + AQ < OQ$$

and we have two sides of the triangle together less than the third.

This will be the case for all values of a, b and c provided that these are sufficiently small, and it is obvious that a similar property will hold for any part of the interval OQ: so that here OQ is a line of *maximum length* in the mathematical sense.

This again is something quite different from what we have in Euclidean geometry and once more our ordinary idea of a straight line breaks down.

The *normality* of lines, etc., exhibits some very curious features in this geometry.

The equations of a line may be put in the form

$$\frac{x - x_1}{l} = \frac{y \quad y_1}{m} = \frac{z - z_1}{n} = \frac{t - t_1}{p}.$$

If
$$\frac{x - x_2}{l'} = \frac{y - y_2}{m'} = \frac{z - z_2}{n'} = \frac{t - t_2}{p'}$$

be a second line, the analytic condition of normality is found to be

$$ll' + mm' + nn' - pp' = 0.$$

If a line be such that

$$l^2 + m^2 + n^2 - p^2 = 0,$$

then analytically, it must be regarded as being *normal to itself*, and this is the type of line for which, as we have already remarked, the conception of length partially breaks down.

Proceeding from the purely analytical standpoint it is easily shown that a line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}=\frac{t}{p}$$

will be normal to a threefold whose equation is

$$lx+my+nz-pt=0.$$

Now any other line through the origin whose equations are

$$\frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'}=\frac{t}{p'}$$

will lie in this threefold provided that

$$ll'+mm'+nn'-pp'=0.$$

If the line (l, m, n, p) be such that

$$l^2+m^2+n^2-p^2=0,$$

then it must itself lie in this threefold to which it is normal.

Thus all lines in such a threefold will be normal to this particular line and, of course, to its parallels, and, if we take $z=0$, we get a plane such that all its lines are normal to a particular set of parallel lines lying in the plane.

Here again is something quite different from what we get in ordinary Euclidean geometry.

It will be found that there are three types of plane and three types of threefold, just as there are three types of line, and the geometrical characters of these are quite distinct from one another.

From the analytical examples which we have given it is evident that this geometry differs in some of the most fundamental respects from that of Euclid and it is clear that, from the pure geometrical standpoint, it must be built up in an entirely different way from that which he employed.

It will be found however that it not only contains Euclidean geometry as an essential part, but that it supplies also what is perhaps the most satisfactory theoretical basis upon which to construct the Euclidean system.

Now we have seen that in this geometry we cannot take a "straight" line as being a *shortest* line and it will be necessary to define it in some other way.

Further, since the coordinate axes in Minkowski's analytical work are supposed to be "straight", we are faced with a serious difficulty even before we are in a position to set up a system of axes.

Moreover the Minkowski axes are supposed to be "normal" to one another and we have seen that there are some rather curious features connected with normality.

We must accordingly build up the subject from the very beginning and must look about us for suitable postulates.

Now in the first place: what is geometry in the general abstract sense?

Geometry has been defined by Whitehead as the "science of cross classification".

The fundamental elements classified are usually called "points" but any entities which satisfy certain postulates may serve the purpose.

Using this definition we may have "geometries" with only a finite number of fundamental elements; but, though interesting as logical curiosities, such systems have no special application in the present state of Science.

The types of geometry with which we are specially concerned when we attempt to map out time and space involve an infinite set of elements forming what is called a "continuum".

The classes of these elements, such as lines, planes, etc., with which we are concerned, are defined by means of certain relations among the elements involved.

In order that the system should be of any use for mapping purposes it is necessary that these relations should have their counterparts in physical space or time.

As to whether these physical counterparts exist or not, the geometry, as a branch of pure mathematics, need not concern itself; but, since the interest of the subject to many persons depends mainly upon the application, we shall devote a little time to a consideration of these matters.

Now in considering the subject of time as it presents itself to our experience there is one very important respect in which it appears to differ from our spacial experience.

Of any two instants which one experiences *in one's own mind* one is *after* the other.

This relation of *after* is what is called an *asymmetrical relation*; by which is meant a relation R such that if B bears the relation R to A then A does not bear the relation R to B.

Thus, in the particular case considered, if B is *after* A, then A is not *after* B.

There are however relations which are *symmetrical*; such, for example, as the relation of *equality*, where if B is *equal to* A then A is *equal to* B.

Now the relation of two points or two particles in space is a *symmetrical relation* and, if A and B be taken as two distinct points, there is no reason why we should say that B is *after* A rather than that A is *after* B.

If we consider points in a straight line it would, of course, be possible to set up some convention according to which we might regard one point as being *after* another; but such convention would be perfectly arbitrary and would not correspond to any natural distinction, as in the case of instants of time in our own consciousness.

Let us consider what actually does hold with regard to the latter.

It is hardly possible to describe what we mean when we use the word *Now*. *Now* singles itself out in the mind and is, as the Germans say, "ausgezeichnet" in some way or other.

Though we speak of *Now* as an *instant*, yet there are innumerable instants, each of which is in its turn a *Now*.

These instants which one experiences in one's own mind have, as already pointed out, an asymmetrical relation one to another; and our very thoughts themselves have a time order, so that we recognize one thought as following *after* another, even if we close our eyes and other channels of sense as far as possible.

We shall not therefore attempt to make any unreal distinction between what is physical and what is mental in respect of the perceptions of a single individual.

These perceptions form a complex picture which is continually changing and, if one splits it up into component parts, one is able to say (at least approximately) that certain events occur at the same instant, while others occur at different instants.

This simultaneity, or lack of it, is an ultimate fact and must be regarded as *absolute*; but we must carefully note what things we are asserting to be simultaneous or otherwise. We are making the assertion about *certain perceptions of a single individual*.

A normal individual who is not a solipsist (and a solipsist could hardly be regarded as a normal individual) believes in the existence of more than his own self and his own perceptions, and one is accustomed

to regard these perceptions, under normal circumstances, as representing things as real as one's self but in some sense *external*.

One naturally thinks of these assumed external events as having a time order, and the first standpoint which one is accustomed to adopt, and which, as a matter of fact, serves for most of the purposes of our daily life, is that these external events occur at the instants at which one perceives them.

More careful observation however convinces one that this cannot be strictly correct, at any rate for all our perceptions, since the perception of an event by one of our senses may be *after* the perception of the same event by another sense.

Thus the visual and auditory perceptions of a blow being struck by a hammer are practically simultaneous when the occurrence is close at hand; but the auditory perception is appreciably *after* the visual perception when the occurrence is at a distance from the observer.

Thus the auditory perception, at any rate, cannot be simultaneous with the distant event and the question naturally arises whether the visual perception is so or not; and, once more, the answer is in the negative.

The first indication that this is the case was obtained by Römer in 1675–6, through observations of the eclipses of Jupiter's satellites; and, though there was a possibility of some other explanation of these observations, such possibility practically vanished when Fizeau, in 1849, was able to test the matter by direct experiment.

Fizeau found that when a flash of light was sent out from the neighbourhood of an observer to a distant mirror which reflected it back to him, the return of the flash occurred at an instant appreciably *after* the instant of its departure.

Thus the instant of one's visual perception of a distant event cannot be identical with the instant at which the event occurs, and we perceive near and distant events simultaneously which certainly do not occur simultaneously.

This fact cannot be ignored if we attempt to correlate astronomical events with one another or with terrestrial ones; although in the ordinary affairs of daily life we can and do ignore it with impunity.

If now we attempt to identify the instant at which a distant event occurs with that of some event near at hand, we find ourselves confronted with very serious difficulties, since this question is intimately bound up with the question of the identification of one and the same

point of space (or the aether of space, if there be such a thing) at different instants of time.

If this latter were possible one would be able to tell when a particle was at "absolute rest" that is to say it would be possible to state that it remained at the same point of space (or of the aether).

If we had an apparatus such as that which Fizeau employed and we could be assured that it remained at "absolute rest" in this sense, and if, for the moment, we neglect any difficulties which there may be in connexion with measurement of space intervals, time intervals or velocity, it would be reasonable to assume that light travels through space (or the aether) with uniform velocity and would take equal intervals of time on its outward and return journeys; so that the instant at the observing station which was midway between the instants of departure and return of the light flash would be identical with the instant of its reflection at the distant mirror.

If, on the other hand, we suppose that observer and apparatus are both in uniform motion, say in the direction of the outward going light, then the mirror would retire in front of the outward going flash, while the observer would advance to meet the returning one, so that the light would have further to travel on its outward than on its return journey, and the instant at the observing station midway between the instants of departure and return would no longer be identical with the instant of reflection.

Now according to the classical mechanics a system of bodies whose centre of inertia is in uniform motion in a straight line is indistinguishable, so far as mechanical effects are concerned, from a similar system whose centre of inertia is at rest.

It is conceivable that some difference might be detected by some optical or electrical device, and many attempts have been made with the object of detecting the motion of the earth through the aether; but none of these attempts has been successful.

Of these attempts, the best known is the celebrated experiment of Michelson and Morley, which consisted in dividing a beam of light into two portions which travelled, the one in one direction and the other in a transverse direction and were reflected back again by mirrors.

If we adopt ordinary ideas for the moment and suppose the light to be propagated with a velocity v through a medium and that the apparatus moves through the medium with velocity u; then it is

easy to calculate the time of the double journey for the two portions of the beam.

For the case of a part of the beam which travels in the direction of motion of the apparatus the time occupied by the double journey is found to be

$$t_1 = \frac{2va_1}{v^2 - u^2},$$

where a_1 is the distance between the point of the apparatus where the beam divides and the corresponding mirror.

If a_2 be the corresponding distance for the case of the transverse portion of the beam, then we can easily show that the time of the double journey should be

$$t_2 = \frac{2a_2}{\sqrt{v^2 - u^2}}.$$

Now if the distances a_1 and a_2 be adjusted so that the times occupied by the two portions of the beam on their journeys are equal, we have

$$\frac{2va_1}{v^2 - u^2} = \frac{2a_2}{\sqrt{v^2 - u^2}}.$$

From this it follows that:

$$a_1 = a_2 \sqrt{1 - \left(\frac{u}{v}\right)^2};$$

so that a_1 should be somewhat less than a_2.

Now the necessary adjustment can be made with extreme accuracy by means of the optical interference bands which are produced and the remarkable fact is observed that, when the whole apparatus is caused to revolve at a uniform slow rate, the one adjustment holds for all positions.

Thus the apparatus gives no evidence of the motion of the earth, although it might be expected to do so.

In order to explain this result the hypothesis was put forward by FitzGerald and Lorentz that the material of the apparatus contracts along the direction of its motion through the aether in the ratio

$$1 : \sqrt{1 - \left(\frac{u}{v}\right)^2}.$$

If however this FitzGerald-Lorentz contraction occurs and bodies change their dimensions in this manner when they move, and if we are unable to detect this motion, what do we really mean by a body remaining of constant length, or of being equal in length to another body?

If such be the case, the distance between the graduations of the most rigid measuring rod will change as the rod is turned in different directions with respect to the earth's motion, and similarly, the shape of the most rigid material triangle will change when we try to superpose it on a material triangle of different orientation.

Admittedly such changes would be very minute under ordinary circumstances, and could generally be neglected; but then, on the other hand, if they are non-existent, how can one explain the null result of the Michelson-Morley experiment, especially in view of the results of some experiments by Lodge which seemed to show that the aether was not carried along by matter moving in the neighbourhood?

Thus we appear to be confronted with formidable difficulties, since, not only can we give no criterion by which to decide that a distant event is simultaneous with one near at hand, but even those physical properties of solid bodies of which use is made in the ordinary measurements of length appear open to question.

The first great steps towards reducing this matter to order were taken by Larmor and Lorentz. These writers showed that the electromagnetic equations could be reduced to the same form for a system moving through an assumed aether as they had for a system "at rest"; and, on the question being raised by Lord Rayleigh in 1902, as to whether rotatory polarisation would be influenced by the earth's motion, and whether such motion would cause double refraction, Larmor was able, from his theory, to predict that no such effects would occur; and this was confirmed by Lord Rayleigh's experiments.

The transformation of the electromagnetic equations involved the introduction of a so-called "local time" and this raises the question as to what is the philosophical significance of this conception.

The view which was put forward by Einstein was that events could be simultaneous for one observer but not simultaneous for another moving with respect to the first.

This view, in my opinion, gives an air of unreality to the external world which cannot be justified; since the events might be the impacts of particles moving with respect to one another, and therefore associated with different "local times", although an impact necessarily involves both particles which impinge and cannot be described without mention of both.

We also think of a definite instant of impact which can be referred to without any mention of "local times" in this sense.

As has already been pointed out, the only simultaneity with which

one is directly acquainted, namely, that of perceptions or ideas in one's own mind, is of an *absolute* character and my contention is that any real simultaneity of external events is also absolute in a similar way.

Let us now examine Einstein's standpoint in order to show in what respect he departs from actually observed or observable facts.

If a flash of light goes out from the neighbourhood of an observer to a distant mirror and is there reflected back to him, then, according to Einstein, the reflection at the distant mirror is simultaneous with an event at the observing station which takes place at the instant midway between the instants of departure and return of the flash of light.

Einstein supposes the instant midway between the instants of departure and return to be determined by means of a clock. Ignoring for the moment the difficulty involved in obtaining an accurate clock; let us consider what this implies.

Let us suppose that to-day I were to observe the outburst of a new star which, in astronomical language, was at a distance of 100 light years, then according to Einstein's view this outburst was simultaneous with terrestrial events which occurred before I was born.

It is evident that this could not be a fact of observation, so far as I am concerned: so that it would be incorrect to speak of such events as *simultaneous for one observer*.

It is frequently asserted that Einstein's theory keeps strictly to observed or observable facts; but here would be a palpable departure from the facts of observation.

The actual observed fact in such a case would be that my *perception* of the outburst would be simultaneous with other experiences of mine occurring to-day.

These are the sort of events which are simultaneous to one observer, and not the occurrence of a distant and a near event, and such simultaneity is *absolute*.

In case it be contended that the above is Einstein's definition of the simultaneity of distant and near events, then our reply is: that if this be so the word simultaneity is being used in two utterly distinct senses in a manner which may lead to great confusion of thought.

In the one case the word is employed correctly to describe something *absolute* while in the other it would be used to describe a mere convention which has not even the merit of being definite without the limitation that the observing station is *unaccelerated* in the interval between the departure and return of the flash of light (or else is accelerated in certain restricted ways).

It is perhaps desirable to point out that it is Einstein's philosophy which I am here attacking and not his mathematics.

This is all the more justifiable, in that the conception of "local time" as a mathematical quantity was introduced, not by Einstein, but by others who did not hold his views.

Even if it were the case that near and distant events were simultaneous, we have, as already pointed out, no means at our disposal of testing this by observation.

A much more notable advance was that made by Minkowski, when he developed a type of four-dimensional analytical geometry in which the change from one set of Time-Space variables to another corresponded to a change of coordinate axes.

The idea of time as a fourth dimension is however much older than Minkowski and dates at least as far back as the time of Lagrange.

The work of Minkowski is purely analytical and does not touch on the difficulties which lie in the application of measurement to time and space intervals and the introduction of a coordinate system.

As regards such measurement; one cannot regard either clocks or measuring rods as satisfactory bases on which to build up a theoretical structure such as we require in this subject.

One knows only too well the difficulty there is in getting clocks to agree with one another; while measuring rods expand or contract in a greater or lesser degree as compared with others.

The existence of a substance such as india-rubber should be sufficient to upset ones trust in measuring rods as ultimate standards; when one considers that it only possesses in an exaggerated degree a property of extensibility common to all solid bodies.

It is not sufficient to say that Einstein's clocks and measuring rods are *ideal* ones: for, before we are in a position to speak of them as being ideal, it is necessary to have some clear conception as to how one could, at least theoretically, recognise ideal clocks or measuring rods in case one were ever sufficiently fortunate as to come across such things; and in case we have this clear conception, it is quite unnecessary, in our theoretical investigations, to introduce clocks or measuring rods at all.

We have in fact a problem to consider regarding the measurement of time and space intervals which may be compared to that which Lord Kelvin set himself in connexion with the measurement of temperature, and which he solved by the invention of the thermodynamic scale.

Now we have seen that in Minkowski's analytical geometry the length of an interval of a line such as the axis of x, y or z is neither a

minimum nor a maximum, while the length of an interval of a line such as the axis of t is actually a mathematical maximum.

Further there are certain lines (which I have called "optical lines") for which the conception of length partially, but not entirely, breaks down.

It thus appears that the conception of length is not at all so simple as is generally supposed and, as a matter of fact, *it is not a fundamental concept at all in Time-Space theory.*

If the measurement of time and space intervals is not fundamental, it may be asked: what is to take its place?

I say that ideas of *order* must take the place of measurement as a theoretical basis; and conceptions of measurement constructed from them.

The process by which this is done is somewhat lengthy, but will be found to shed an important light on the seeming paradoxes above mentioned.

In constructing this system it is necessary to modify certain currently accepted notions, but the modifications required all appear to be capable of justification and the structure when completed will be found closely to resemble our ordinary conceptions.

We shall regard an instant as a fundamental concept which, for present purposes, it is unnecessary further to analyse, and shall consider the relations of order among the instants of which I am directly conscious.

Thus for such instants we find the following properties:

(1) If an instant B be *after* an instant A, then the instant A is not *after* the instant B, and is said to be *before* it.

(2) If A be any instant, there is at least one instant which is *after* A and also at least one instant which is *before* A.

(3) If an instant B be *after* an instant A, there is at least one instant which is both *after* A and *before* B.

(4) If an instant B be *after* an instant A and an instant C be *after* the instant B, then the instant C is *after* the instant A.

(5_0) If an instant A be neither *before* nor *after* an instant B, the instant A is identical with the instant B.

Now it appears to have been too hastily assumed, because the set of instants of which a single individual is directly conscious possess all

these properties, that therefore they must hold in respect of all instants throughout the universe.

It would appear that people in general have been making a somewhat similar blunder to the one ascribed to Sir Boyle Roche; who is alleged to have asserted in a speech, that he could not have been in two places at once unless he were a bird.

They have assumed that an instant, like Sir Boyle Roche's bird, could be in two places at once, and, in consequence, they have found extreme difficulty in identifying it as one and the same instant in the two places.

Had Sir Boyle Roche pursued the subject further, he might perhaps have arrived at a form of relativity theory whereby a bird might be simultaneously in two places to one observer, but not to another. However he got sufficiently far to achieve immortal fame.

I, however, venture to dissent from the generally accepted view that an instant can be in two places at once and, while still regarding postulate (5_0) as holding for the set of instants of which any one individual is directly conscious, or which any single particle occupies, I propose to reject it for the universe in general and to substitute for it the following:

(5) If A be any instant, there is at least one instant distinct from A which is neither *before* nor *after* A.

If I am directly conscious of the instant A then any instant such as that here postulated will be one of which I am not directly conscious, but only indirectly apprehend, and which is, as I say, *elsewhere*.

The other four postulates are however to be regarded as holding in general and not merely for a single individual or a single particle.

Since we are able to distinguish an instant elsewhere in terms of *before* and *after* relations, it is unnecessary to have any separate concept of *space*; since the geometrical properties of space can be expressed in terms of these relations; although, of course, this involves an elaborate logical analysis.

While the set of instants of which any single individual is conscious or which any single particle of matter occupies have a linear order, the set of instants for the universe in general appear to have what I have called a *Conical Order*.

I have given it this name because it may be illustrated by means of ordinary geometric cones: while it contains within itself the possibility

of defining particular sets of instants having a simple linear order such as that with which each of us is familiar.

It is to be recollected that this illustration is given merely as a mental aid to enable us to grasp a certain set of abstract relations, just as figures are an aid in doing geometry; but, as in the latter case, everything which we may introduce incidentally and which cannot be described in terms of the abstract relations is to be ignored.

Without some such picture it would be rather difficult for most people to retain all the relations in mind in complicated cases, and moreover, as in ordinary geometry, a diagram will often suggest that certain theorems may hold and may also suggest methods of proof.

Let us suppose that we have a system of right circular cones of equal angle and with their axes all parallel (or identical). We shall suppose each cone to terminate at the vertex, which however is to be regarded as a point of the cone.

We shall call such a cone having its opening pointed in one direction (say upwards) an α cone and one having the opening pointed in the opposite direction a β cone.

Corresponding to any point of space we shall have an α cone and a β cone having the point as a common vertex.

Now it is possible by using such cones and making a convention with respect to the use of the words *before* and *after* to set up a type of order of the points of space.

For the purposes of this illustration we shall make the convention that if A be the common vertex of such a pair of α and β cones, then any point will be said to be *after* A provided that it is distinct from A and lies either on or inside the α cone of A and will be said to be *before* A provided that it is distinct from A and lies either on or inside the β cone of A, but not otherwise.

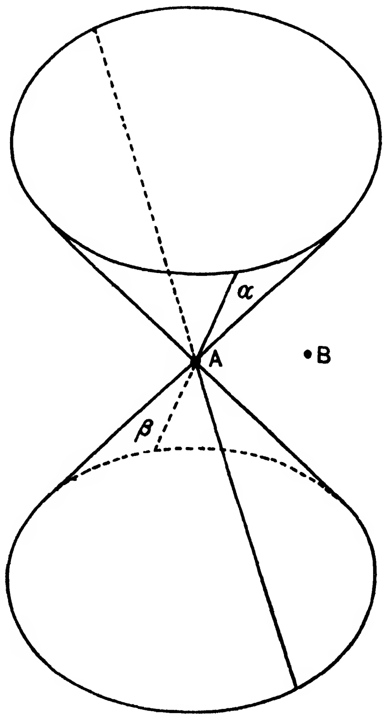


Fig. 1.

Thus any point which is either identical with A or else lies outside both the cones α and β will be neither *before* nor *after* A.

It is easy to see that postulates (1), (2), (3), (4) and (5) hold generally in this illustration substituting points for instants, but that (5_0) only holds for certain sets of points forming lines straight or curved.

We may, by a study of such models, ascertain various other *before* and *after* relations which hold among the points and we can then reverse the process by taking the *before* and *after* relations as starting point instead of the cones, and propositions expressed in terms of these relations as postulates, and can, in this way, build up a system of geometrical relations very closely analogous to, but not quite identical with, those from which we started out and which involve nothing except what can be expressed in terms of the *before* and *after* relations.

In this way we are able to define what I call α and β sub-sets, which have many, although not all the properties which we assigned to the α and β cones, and can gradually, step by step, build up a system of geometry which is equivalent to the analytical system of Minkowski.

Our model is only three-dimensional, while the geometry of Minkowski is four-dimensional; but, in spite of this, most of our postulates may be represented in three dimensions, and, in fact, there are only two which cannot. One of these introduces a fourth dimension, while the other limits the number of dimensions to four.

We could extend the system to a larger number of dimensions if required, but we do not propose to do so in this work.

If we consider straight lines in our model passing through the point A, we observe that such a line may be of three distinct types. The first type falls within the cones α and β; the second type forms a generator of these cones; while the third type falls outside the cones.

The first and second types have this in common, that, if we consider two distinct points lying in either type of line, one is *after* the other; while if we consider any two distinct points in the third type of line the one is neither *before* nor *after* the other.

Again, if we consider planes through the point A we see that they too may be of three distinct types. The first type intersects the cones α and β in two generators; the second type touches the cones along a generator; while the third type has no point in common with the cones except the point A.

Similarly there are three types of threefold, but in order to represent them we should require a four-dimensional model.

These different types of line, plane and threefold may all be defined in terms of *before* and *after* relations.

In one important respect however our model differs from our logical constructions. Equal lengths in the model do not, in general, represent equal lengths in our geometry: the latter being defined by a certain analysable similarity of *before* and *after* relations.

The reason why there is this difference between the model and the system of geometry which we build up, is that the model has already got a system of measurement imposed upon it, owing to the fact that it is constructed in ordinary three-dimensional space, and so involves more than the mere *before* and *after* relations which it was designed to illustrate.

Finally we are able to introduce coordinates and the system is then seen to be equivalent to the analytical geometry of Minkowski.

In such a system as he employed, one coordinate is measured along a line corresponding to one lying within the cones in the model (and which we shall call an inertia line), and represents what clocks purport to measure. The other three coordinates are measured along lines corresponding to those lying outside the cones (or separation lines) and these represent what we call spacial distances.

The four coordinate axes in this system are all normal to one another (normality being also capable of definition in terms of *before* and *after*), but, if we do not insist on normality, it is possible to introduce a symmetrical system of coordinates in which all four are measured along lines of the same type.

Now, as the *before* and *after* relations from which our whole theory is built up have a temporal significance, we appear to have absorbed the theory of space in a theory of time, in which instants have a conical order instead of the purely linear order which they are generally regarded as having.

An instant for the universe in general is identified by four coordinates in this theory instead of merely one coordinate as is generally assumed.

An instant is localised and does not range all over the universe like Sir Boyle Roche's bird: so that *the present instant does not extend beyond here, and the only really simultaneous events are events which occur at the same place.*

In Minkowski's system of coordinates the so-called "local time" is merely the value of that particular coordinate which is measured along an inertia line.

If we take a second normal coordinate system in which the inertia

axis is not parallel to the former, we have one which is appropriate to a material system which is moving uniformly with respect to the first, and we have a different "local time".

An inertia line is the time path of an unaccelerated particle, and, since it is defined in terms of *before* and *after* relations, we are able to say in terms of these relations what we mean by a particle being *unaccelerated*.

We can however assign no meaning to a particle being at "absolute rest" since, in this geometry, any inertia line is exactly on a par with any other one.

Thus instead of regarding ourselves as, so to speak, swimming along in an ocean of space (as we usually do), we are to think of ourselves rather as somehow pursuing a course in an ocean of time; while *spacial relations are to be regarded as the manifestation of the fact that the elements of time form a system in conical order: a conception which may be analysed in terms of the relations of after and before.*

It should be noted that the fundamental relation of *after* which serves as a basis for Time-Space theory is simpler than the relation which geometers are accustomed to make use of in building up ordinary three-dimensional geometry.

The basic relation which they employ is generally the relation of *between*: one point being linearly between two others. This is a relation involving three terms, whereas the *after* relation is one involving only two.

It will appear in the course of this work that a relation of *linearly between* may be defined in terms of *before* and *after* relations for the case of three elements in a separation line; although no one of these three elements is either *before* or *after* either of the other two.

One could scarcely hope to do the converse of this, that is to say, to define an asymmetrical relation of two elements in terms of one like *linearly between* involving three.

It is true that spacial models involving cones may be used to illustrate graphically various postulates employed in our geometry, but this can only be done by means of an arbitrary convention as to what should represent *after* and what *before*.

This convention might have been reversed without affecting the usefulness of the representation; but, by no stretch of the imagination can one (so far as I can see) reverse the time relations of *before* and *after* which one perceives directly in one's own mind. *The thought process is essentially an irreversible one.*

Another interesting point to note is that whereas, on the one hand, if ordinary geometry is built up from the *between* relation, the theory of *congruence* appears as something extraneous grafted on to an otherwise complete scheme; on the other hand, if the *before* and *after* relations are used as a basis, *congruence* appears as an intrinsic part of the subject.

Let us now consider what is the physical peculiarity of the time relations of *before* and *after* which gives them their asymmetrical character.

One thing seems clear: If I at the instant A can produce any effect, however slight, at a distinct instant B, then this is sufficient to imply that B is *after* A.

A present action of mine may produce some effect to-morrow, but nothing which I may do now can have any effect on what occurred yesterday.

It appears to me that we have here the essential feature of what we mean when we use the word *after*, and that the abstract power of a person or living being at the instant A to produce an effect at a distinct instant B is not merely a *sufficient* but also a *necessary* condition that B is *after* A.

If however a person at the instant A cannot produce an effect at the instant B, it does not follow that B is *before* A.

In order that this should be so it would be necessary that a person at B should be able to produce an effect at A; since *before* and *after* are converse relations.

Thus the significance of an instant A being neither *before* nor *after* a distinct instant B, is that a person at A should be unable to produce any effect at B and a person at B should be unable to produce any effect at A.

We shall have to give some further consideration to this idea of *possibility of producing an effect*; but, before doing so, we shall first consider the physical circumstances under which one instant is neither *before* nor *after* another.

In the first place it is to be observed that, regarded from the standpoint of pure mathematics, the system of geometry which we are about to develop only presupposes that there should be a set of elements which are related in a certain way which can be analysed in terms of a certain asymmetrical relation.

In our attempt to apply this, we identify an element with an *instant*, and the asymmetrical relation with the physical relation of *after*.

The suitability or otherwise of this abstract geometry for describing actual time and space relations is dependent upon the degree of accuracy with which the various postulates of the geometry correspond with various physical facts.

Now it appears to be possible to establish a very close, although perhaps not an exact, correspondence of this sort by means of the physical properties of light.

Let P and Q be two separate and distinct particles and let a flash of light be sent out from P at the instant A so as to arrive directly at Q at the instant B, then, according to our interpretation of *after*, B is *after* A.

Further, there are strong physical grounds for believing, at any rate in the absence of appreciable quantities of matter, that light supplies a criterion which, with the meaning we have above ascribed to *after*, enables us to say that B is the first instant at Q which is *after* A and that A is the last instant at P which is *before* B.

It will be observed that no mention of *velocity* is made in this statement but merely the *before* and *after* relations.

The conception of velocity involves the conception of *measurement* of space and time intervals and these are supposed to be not yet defined.

Let us suppose next that the light flash is reflected directly back from Q to P and that it arrives there at the instant C, then, if the view we have mentioned be correct, any instant at P which is *after* A and *before* C will be neither *before* nor *after* B.

Now Fizeau's apparatus is an arrangement in which this is practically carried out: so that we can say that *any instant at the sending apparatus which is* after *the instant of departure of a flash of light and* before *the instant of its return is neither* before *nor* after *the instant of reflection at the distant mirror*.

It is possible that the analytical geometry of Minkowski, with these optical interpretations of our postulates, gives only an approximate, although under ordinary circumstances a very closely approximate representation of time and space relations, and this is the view now held by Einstein and others; but even so, it does not follow that with some slightly different interpretation it may not be exact.

But, as we shall see, the *before* and *after* relations enable us to define equality of intervals in Minkowski's geometry, and, however the Time-Space universe may be constituted, these relations certainly have some physical significance; so that there can be little doubt that

they must play just as important a rôle in the foundations of any generalised theory as they do in the simple one.

In fact the Minkowski theory might perhaps be regarded as giving the constitution of Time-Space provided that we do not consider too large a portion of it, while the so-called generalised theory would be the sort of thing we should get provided, in our model, the cones, instead of being all similar and similarly situated, varied from one point to another.

I ought perhaps to remark that any proper quadric cone would serve equally well to illustrate all our postulates and it is only for the sake of simplicity that I supposed the cones to be right circular ones.

Before one is in a position to set up any type of coordinates it is fairly evident that one must, either tacitly or explicitly, make use of considerations of order, if these coordinates are to have any sort of system about them, and the *before* and *after* relations appear to have the requisite fundamental character to supply this.

The view that time relations are fundamental appears to have an important bearing on what Professor William James called the theory of a "block universe": by which name he referred to the theory that the universe is something like a cinematograph film in which the photographs have already been taken and which is merely in process of being exhibited to us.

Most writers on this subject treat time as if it were merely a fourth dimension of space: an attitude which encourages one to favour the "block universe" idea.

When instead, we regard *before* and *after* relations as fundamental, and analyse spacial relations up in terms of these, the whole subject appears in a very different light and the "block universe" theory does not commend itself so strongly.

If the universe were in this way like a cinematograph film which is merely being displayed before us, then its innumerable details must have been fixed through all eternity and there would be complete determinism as to the future.

But have we really any grounds for thinking that the universe is of this nature: or, reverting to the cinematograph analogy, is it any simpler to suppose that the film has already been taken than to suppose that the play is in process of being acted?

If the *after* relation has the significance which I suggested and if what we call time and space may be analysed in terms of *before* and

after then it would seem that instead of having grounds for belief in a "block universe" we have actually got grounds for an opposite view.

It seems therefore that the question turns on the significance of the *after* relation and its asymmetric character.

It is interesting to note that recently, on quite different grounds, some physicists are coming round to the view that the universe is not strictly deterministic.

Scientific predictions as to future events are made on the assumption that certain uniformities will continue.

If they do continue the prediction may be a logical consequence of their doing so, but, if the uniformities do not continue, the conclusion may be unwarranted.

The continuance of the uniformities is only an assumption for which we have no absolute guarantee, and, should they cease, no promise is broken, since none was ever made. A departure from uniformity initiated at an instant A may extend to an instant B which is *after* A; and this would be an *effect* at B of the departure from uniformity initiated at A.

All applied mathematics becomes pure mathematics when we get away from our fundamental assumptions and begin to draw logical conclusions from them.

Now I have ascribed certain characteristics to instants and to *before* and *after* relations which may or may not be strictly correct, but which serve as the basis by means of which one may apply a certain type of pure geometry to map out time and spacial relations.

The geometry, as I have already pointed out, is a logical structure built up from certain postulates which I shall formulate.

As a logical structure a geometry may have more than one application, as for instance, ordinary Euclidean plane geometry might be taken primarily as applying to figures on what we call a plane and again to geodesic lines drawn on a developable surface.

For the purposes of physical science, however, it is not sufficient merely that we should say, for instance, that *there are* such things as "straight lines" or that *there are* lengths which are equal, but it is necessary to have criteria by which we can say (at least approximately) "*here are* points which lie in a straight line" and "*here is* a length which is equal to yonder length".

In other words we must have more or less clear ideas of the physical things to which we apply our abstract theory.

The abstract theory itself does not require this, but the physical

application does; and for this reason, I have tried to make clear the sort of physical meaning which I ascribe to the notions of an instant, the *before* and *after* relations and the criteria given by light flashes.

If we should discover, for instance, that the formal properties which we provisionally ascribe to light actually hold for some other influence; then the geometry which I propose to develop would apply with this new interpretation of its postulates.

Now I have made use of ordinary geometric cones in order to enable us to form a concrete picture of what I mean by "*conical order*", but the idea of conical order is not at all dependent upon this graphic representation, but is built up by a rather lengthy piece of reasoning from the asymmetrical relations which I denote by the words *before* and *after*.

The representation by means of cones may be compared to the rough scaffolding used in the erection of a building which is removed when the building is complete and its component parts in position.

We must, however, be certain that the building is not supported by the scaffolding, or it will not be able to stand alone.

In order to make sure of this in our theory, great care must be taken not to take things for granted because they hold in our models.

In the first place we are not at liberty to introduce coordinates except for scaffolding purposes until we have defined them. Neither are we at liberty to speak of "velocity" except for scaffolding purposes till its meaning is defined.

Moreover in the actual proof of theorems we must not employ the ideas of equality of lengths or angles until these ideas are seen to be definable in terms of *before* and *after* relations.

We may however, and actually do, make use of such non-permissible ideas in our graphic representation.

Thus in the models we supposed the cones to have their axes parallel (or identical) and to have equal vertical angles, and neither the idea of *cone*, of *parallel*, of *axis*, of *angle*, nor of *equal* has been analysed in terms of *before* and *after* and therefore must be excluded in defining the α and β sub-sets, which are the names which I shall hereafter apply to the entities corresponding to the α and β cones.

The *before* and *after* relations are converse asymmetrical relations and either may be defined in terms of the other; so that it is a matter of indifference which of them we take as fundamental.

I actually take the relation of *after* as fundamental and define *before* in terms of it.

As regards the postulates which are expressed in terms of these relations, they generally consist of two parts (marked *a* and *b*) in which the *before* and *after* relations are interchanged.

In some of the postulates, however, the one part follows from the other on account of the mutual relations of *after* and *before*: while in some others the *before* and *after* relations are involved symmetrically.

We shall now proceed with the formal development of the subject.

CONICAL ORDER

We shall suppose that we have a set of elements and that certain of these elements stand in a relation to certain other elements of the set which we denote by saying that one element is *after* another.

We shall further assume the following conditions:

POSTULATE I. **If an element B be after an element A, then the element A is not after the element B.**

This is merely the condition that *after* should be an asymmetrical relation. If an element B be *after* an element A, it follows directly from Post. I that A and B must be distinct elements, for, if we substitute A for B in the postulate, it becomes self-contradictory.

Definition. If an element B be *after* an element A, then the element A will be said to be *before* the element B.

POSTULATE II. (*a*) **If A be any element, there is at least one element which is after A.**

(*b*) **If A be any element, there is at least one element which is before A.**

POSTULATE III. **If an element B be after an element A, and if an element C be after the element B, the element C is after the element A.**

POSTULATE IV. **If an element B be after an element A, there is at least one element which is both after A and before B.**

POSTULATE V. **If A be any element, there is at least one other element distinct from A, which is neither before nor after A.**

POSTULATE VI. (*a*) **If A and B be two distinct elements, one of which is neither before nor after the other, there is at least one element which is after both A and B, but is not after any other element which is after both A and B.**

(*b*) **If A and B be two distinct elements, one of which is neither after nor before the other, there is at least one element which is before both A and B, but is not before any other element which is before both A and B.**

Definition. (*a*) If A be any element of the set, then an element X will be said to be a member of the α sub-set of A provided X is either

identical with A, or else provided there exists at least one element Y distinct from A and neither *before* nor *after* A and such that X is *after* both A and Y but is not *after* any other element which is *after* both A and Y.

(*b*) If A be any element of the set, then an element X will be said to be a member of the β sub-set of A provided X is either identical with A, or else provided there exists at least one element Y distinct from A and neither *after* nor *before* A and such that X is *before* both A and Y but is not *before* any other element which is *before* both A and Y.

If A be any element then, by Post. V, there is at least one other element distinct from A which is neither *before* nor *after* A and so it follows directly by Post. VI (*a*) that there is at least one other element besides A which is a member of the α sub-set of A.

Similarly, by Post. VI (*b*), there is at least one other element besides A which is a member of the β sub-set of A.

Notation. We shall denote by α_1 and β_1 the sub-sets corresponding to an element A_1, and by α_2 and β_2 those corresponding to an element A_2, etc.

POSTULATE VII. (*a*) **If A_1 and A_2 be elements and if A_2 be a member of α_1, then A_1 is a member of β_2.**

(*b*) **If A_1 and A_2 be elements and if A_2 be a member of β_1, then A_1 is a member of α_2.**

POSTULATE VIII. (*a*) **If A_1 be any element and A_2 be any other element in α_1, there is at least one other element distinct from A_2 which is a member both of α_1 and of α_2.**

(*b*) **If A_1 be any element and A_2 be any other element in β_1, there is at least one other element distinct from A_2 which is a member both of β_1 and of β_2.**

THEOREM 1

If A_1 be any element and A_2 be any other element in α_1, then any element A_3 which is both after *A_1 and* before *A_2, must be a member both of α_1 and β_2.*

By the definition of a member of the sub-set α_1 there exists at least one element, say A_4, distinct from A_1 and neither *before* nor *after* A_1 and such that A_2 is *after* both A_1 and A_4 but is not *after* any other element which is *after* both A_1 and A_4.

Then A_4 cannot be *after* A_3, for if it were then, by Post. III, A_4 would be *after* A_1 contrary to hypothesis.

Further A_4 cannot be identical with A_3, for then again we should have A_4 *after* A_1 contrary to hypothesis.

Again A_4 cannot be *before* A_3 for then we should have A_2 *after* the element A_3 which would be *after* both A_1 and A_4 contrary to the hypothesis that A_2 is *after* both A_1 and A_4 but not *after* any other element which is *after* both A_1 and A_4.

Thus A_4 is distinct from A_3 and is neither *before* nor *after* A_3.

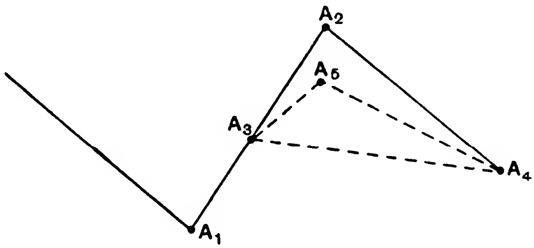


Fig. 2.

Now A_2 cannot be *after* any other element which is *after* both A_3 and A_4, for if A_5 were such an element it would follow by Post. III that since A_3 is *after* A_1 we should have A_5 *after* A_1.

Thus we should have A_2 *after* A_5 which would be *after* both A_1 and A_4 contrary to hypothesis.

Thus no such element as A_5 can exist and so A_2 satisfies the definition of being a member of α_3.

Thus by Post. VII (a) it follows that A_3 is a member of β_2.

Again by Post. VII (a) since A_2 is a member of α_1 it follows that A_1 is a member of β_2, and so by a similar method we may prove that A_3 is a member of α_1. Thus the theorem is proved.

Theorem 2

(a) *If A_1 be any element and A_2 be any other element in α_1, there is at least one other element in α_1 distinct from A_2 which is neither* before *nor* after A_2.

Since A_2 is a member of α_1 it follows by Post. VII (a) that A_1 is a member of β_2.

Thus there exists at least one other element, say A_3, distinct from A_2 and neither *before* nor *after* A_2 and such that A_1 is *before* A_2 and A_3, but is not *before* any other element which is *before* both A_2 and A_3.

Thus A_1 satisfies the definition of being a member both of β_2 and β_3 and so, by Post. VII (b), A_3 is also a member of α_1. Thus since A_3 is distinct from A_2 and neither *before* nor *after* A_2, the theorem is proved.

(*b*) *If* A_1 *be any element and* A_2 *be any other element in* β_1, *there is at least one other element in* β_1 *distinct from* A_2 *which is neither* after *nor* before A_2.

Definition. If A_1 be any element and A_2 be any other element in α_1, the *optical line* A_1A_2 is defined as the aggregate of all elements which lie either

(1) both in α_1 and α_2,

or (2) both in α_1 and β_2,

or (3) both in β_1 and β_2.

THEOREM 3

(*a*) *If a be any optical line, there exists at least one element which is not an element of the optical line, but is* before *some element of it.*

If A_1 be any element and A_2 be any other element in α_1 then, by Post. VII (*a*), A_1 is a member of β_2.

Thus by Theorem 2 (*b*) there is at least one other element in β_2 distinct from A_1 which is neither *after* nor *before* A_1.

Call such an element A_3.

Then since A_3 is in β_2 and distinct from A_2 it is *before* A_2.

But A_3 cannot lie in the optical line A_1A_2, for by the definition of the optical line A_1A_2, in order to lie in it A_3 would require to lie also either in α_1 or β_1.

But if A_3 should lie in α_1 it would be either *after* A_1 or identical with A_1, while if it should lie in β_1 it would be either *before* A_1 or identical with A_1.

But A_3 is distinct from A_1 and is neither *after* nor *before* A_1 and therefore does not lie in the optical line A_1A_2, although it is *before* A_2 an element of it.

(*b*) *If a be any optical line, there exists at least one element which is not an element of the optical line, but is* after *some element of it.*

POSTULATE IX. (*a*) **If *a* be an optical line and if A_1 be any element which is not in the optical line but before some element of it, there is one single element which is an element both of the optical line *a* and the sub-set α_1.**

(*b*) **If *a* be an optical line and if A_1 be any element which is not in the optical line but after some element of it, there is one single element which is an element both of the optical line *a* and the sub-set β_1.**

Theorem 4

(*a*) *If* A_1 *be any element there is at least one other element which is* after A_1 *but is not a member of the sub-set* α_1.

Let A_2 be any other member of the sub-set α_1 distinct from A_1.

Then A_2 is *after* A_1 and so by Post. IV there is at least one element, say A_3, which is both *after* A_1 and *before* A_2.

By Theorem 1 A_3 is a member both of α_1 and of β_2 and is therefore an element of the optical line A_1A_2.

But since A_3 is a member of β_2 it follows that A_2 is a member of α_3 and so by Theorem 2 there is at least one other element in α_3 distinct from A_2 which is neither *before* nor *after* A_2.

Let A_4 be such an element.

Then since A_4 is neither *before* nor *after* A_2 it cannot be a member either of β_2 or α_2 and so A_4 is not an element of the optical line A_1A_2 although it is *after* A_3 an element of it.

But since A_4 is a member of α_3 it follows by Post. VII (*a*) that A_3 is a member of the sub-set β_4.

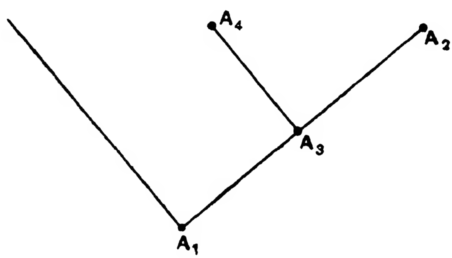


Fig. 3.

Thus A_3 is the *one single element* which by Post. IX (*b*) is an element both of the optical line and the sub-set β_4.

But A_4 cannot be a member of α_1, for then A_1 would be a member of β_4 and so A_1 would be a second element common to the optical line A_1A_2 and the sub-set β_4, which is impossible by Post. IX (*b*).

Further, A_4 is *after* A_3 and A_3 is *after* A_1 and therefore A_4 is *after* A_1.

Thus A_4 is *after* A_1 but is not a member of the sub-set α_1.

(*b*) *If* A_1 *be any element there is at least one other element which is* before A_1 *but is not a member of the sub-set* β_1.

Theorem 5

If A_1 *be any element and* A_2 *be any other element which is* after A_1, *there is at least one other distinct element which is a member of both* α_1 *and* β_2.

Two cases arise: (1) A_2 may be a member of α_1 or (2) A_2 may not be a member of α_1.

If A_2 is a member of α_1 then by Post. IV there is at least one element which is both *after* A_1 and *before* A_2, and by Theorem 1 such an element is a member both of α_1 and β_2.

Thus case (1) is proved.

Suppose next that A_2 is not a member of α_1 and let A_3 be any element of α_2 distinct from A_2.

Then the optical line A_2A_3 which for brevity we may call a, consists of the aggregate of all elements which lie either

(1) both in α_2 and α_3,

or (2) both in α_2 and β_3,

or (3) both in β_2 and β_3.

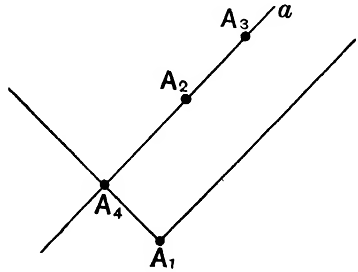


Fig. 4.

Since A_2 is not a member of α_1 it follows that A_1 is not a member of β_2 and so, since A_1 is *before* A_2 it follows that A_1 is not an element of the optical line a.

Then by Post. IX (a) since A_1 is not an element of the optical line a but is *before* an element of it, it follows that there is one single element which is an element both of the optical line a and the sub-set α_1.

Let A_4 be this element.

Then since we have supposed that A_2 is not a member of α_1 it follows that A_4 is not identical with A_2.

Further, A_4 cannot be *after* A_2 for then we should have A_2 *after* A_1 and *before* A_4 and so by Theorem 1 we should have A_2 a member of α_1 contrary to hypothesis.

Thus A_4 cannot be a member of α_2 and therefore since it is an element of the optical line a it must be a member of β_2 and β_3.

Thus the element A_4 is a member of both α_1 and β_2 and so the theorem is proved.

Theorem 6

(a) *If A_1 be any element and A_2 be any other element in α_1, while A_3 is an element distinct from A_2, which is a member both of α_1 and of α_2, then there is at least one other element which is a member of α_1, of α_2 and of α_3.*

By Post. VIII (a) since A_3 is an element of α_2 distinct from A_2 there is at least one other element distinct from A_3 which is a member both

of α_2 and of α_3. Call such an element A_4. Then since A_4 is in α_3 and distinct from A_3 it is *after* A_3.

Thus A_4 is *after* an element of the optical line A_1A_2.

But A_4 is a member of α_2 and also of α_3 and so by Post. VII (a) A_2 and A_3 are each members of β_4.

Now if A_4 were not in the optical line A_1A_2 it would follow by Post. IX (b) that there was *one single element* which was an element both of the optical line and the sub-set β_4.

There are however *at least two elements* A_2 and A_3 with this property and so A_4 must be in the optical line A_1A_2.

Also since A_4 is in α_2 it must also be in α_1 from the definition of the optical line.

Thus A_4 is a member of α_1, of α_2 and of α_3.

(b) *If A_1 be any element and A_2 be any other element in β_1, while A_3 is an element distinct from A_2, which is a member both of β_1 and of β_2, then there is at least one other element which is a member of β_1, of β_2 and of β_3.*

THEOREM 7

(a) *If X be any element of an optical line there is at least one element of the optical line which is* after *X.*

Let the optical line be defined by any element A_1 and another element A_2 in α_1. Then X may lie either

(1) both in α_1 and α_2,

or (2) both in α_1 and β_2,

or (3) both in β_1 and β_2.

If X be not identical with A_2, then in cases (2) and (3) since X lies in β_2, the element A_2 is *after* X.

If X be identical with A_2, then by Post. VIII (a) there is at least one other element distinct from A_2 which is a member both of α_1 and of α_2 and is therefore an element of the optical line.

Since such an element is not identical with A_2 it must be *after* A_2; that is to say it must be *after* X.

Next suppose X is in both α_1 and α_2 and is distinct from A_2.

It follows by Theorem 6 (a) that there is at least one *other* element which is a member of α_1 and α_2 and of the α sub-set of X.

Since such an element is not identical with X and lies in the α sub-set of X it must be *after* X.

Further since it is an element both of α_1 and of α_2 it lies in the optical

line. Thus in all cases there is at least one element of the optical line which is *after* X.

(*b*) *If X be any element of an optical line there is at least one element of the optical line which is* before X.

THEOREM 8

(*a*) *If A_1 be any element and A_2 be any other element in α_1, and if A_3 and A_4 be other distinct elements which are members of both α_1 and α_2, one of the two elements A_3 and A_4 is in the α sub-set of the other.*

Since A_3 is in α_2 and distinct from A_2 therefore A_2 and A_3 define an optical line. Further since A_2 and A_3 both lie in α_1 therefore A_1 lies in both β_2 and β_3.

Thus A_1 is an element of the optical line A_2A_3.

But A_4, since it is a member of α_1 and not identical with A_1, is *after* A_1.

That is to say, it is *after* an element of the optical line A_2A_3.

If then A_4 were not an element of the optical line A_2A_3 there would, by Post. IX (*b*), be *one single element* which would be an element both of the optical line A_2A_3 and the sub-set β_4.

But A_4 is a member both of α_1 and of α_2 and so both A_1 and A_2 are members of β_4.

Thus since A_1 and A_2 are two distinct elements of the optical line A_2A_3 it follows that A_4 must be an element of the same optical line.

But A_4 is a member of α_2 and therefore by the definition of the optical line A_4 must be either a member of α_3 or of β_3.

If A_4 be a member of β_3, then we should have A_3 a member of α_4.

Thus one of the two elements A_3 and A_4 lies in the α sub-set of the other.

It also follows since A_3 and A_4 are supposed to be distinct, that the one is *after* the other.

(*b*) *If A_1 be any element and A_2 be any* other *element in β_1, and if A_3 and A_4 be other distinct elements which are members of both β_1 and β_2, one of the two elements A_3 and A_4 is in the β sub-set of the other.*

It also follows since A_3 and A_4 are supposed to be distinct that the one is *before* the other.

THEOREM 9

If a pair of elements be in an optical line defined by another pair of elements, then one of the first pair is in the α sub-set of the other.

Consider the optical line defined by the element A_1 and another element A_2 in α_1. Suppose now in the first place that we have an element A_3 distinct from A_1 and A_2 and lying in the optical line.

Then by the definition of an optical line A_3 may be

(1) both in α_1 and α_2,

or (2) both in α_1 and β_2,

or (3) both in β_1 and β_2.

Thus if A_1 and A_3 be taken as a pair of elements in the optical line defined by A_1 and A_2, we have in the first and second cases A_3 is in α_1, while in the third we have A_3 in β_1 and consequently A_1 in α_3. Thus one of the pair A_1, A_3 is in the α sub-set of the other.

Again if A_2 and A_3 be taken as a pair of elements in the optical line defined by A_1 and A_2, we have in the first case A_3 is in α_2, while in the second and third we have A_3 in β_2 and consequently A_2 in α_3. Thus one of the pair A_2, A_3 is in the α sub-set of the other.

Next suppose that we have another element A_4 lying in the optical line and distinct from A_1, A_2 and A_3.

Then there are the following possibilities:

$$A_3 \text{ both in } \alpha_1 \text{ and } \alpha_2 \text{ with } \begin{cases} \phantom{\text{or }} A_4 \text{ both in } \alpha_1 \text{ and } \alpha_2 \ldots\ldots(1), \\ \text{or } A_4 \text{ both in } \alpha_1 \text{ and } \beta_2 \ldots\ldots(2), \\ \text{or } A_4 \text{ both in } \beta_1 \text{ and } \beta_2 \ldots\ldots(3). \end{cases}$$

$$A_3 \text{ both in } \alpha_1 \text{ and } \beta_2 \text{ with } \begin{cases} \phantom{\text{or }} A_4 \text{ both in } \alpha_1 \text{ and } \alpha_2 \ldots\ldots(4), \\ \text{or } A_4 \text{ both in } \alpha_1 \text{ and } \beta_2 \ldots\ldots(5), \\ \text{or } A_4 \text{ both in } \beta_1 \text{ and } \beta_2 \ldots\ldots(6). \end{cases}$$

$$A_3 \text{ both in } \beta_1 \text{ and } \beta_2 \text{ with } \begin{cases} \phantom{\text{or }} A_4 \text{ both in } \alpha_1 \text{ and } \alpha_2 \ldots\ldots(7), \\ \text{or } A_4 \text{ both in } \alpha_1 \text{ and } \beta_2 \ldots\ldots(8), \\ \text{or } A_4 \text{ both in } \beta_1 \text{ and } \beta_2 \ldots\ldots(9). \end{cases}$$

In case (1) by Theorem 8 (*a*) one of the two elements A_3 and A_4 is in the α sub-set of the other. Similarly in case (9) by Theorem 8 (*b*) one of the two elements A_3 and A_4 is in the β sub-set of the other, and therefore by Post. VII (*b*) one of them is in the α sub-set of the other.

Consider next case (2).

Since A_4 is in α_1 and distinct from A_1 it follows that A_4 is *after* A_1.

Further, since A_4 is in β_2 and distinct from A_2 we have A_2 *after* A_4, and since A_3 is in α_2 and distinct from A_2 we have A_3 *after* A_2.

Thus by Post. III A_3 is *after* A_4.

But, since A_3 is in α_1, it follows by Theorem 1 that A_4 is in β_3 and consequently A_3 lies in α_4.

Similarly in case (4) we may prove that A_4 must lie in α_3.

In an analogous manner in case (8) since A_4 is in β_2 and distinct from A_2 we have A_4 is *before* A_2.

Further, since A_4 is in α_1 and distinct from A_1, we have A_4 is *after* A_1, and since A_3 is in β_1 and distinct from A_1 we have A_1 is *after* A_3 and so, by Post. III, A_4 is *after* A_3.

But since A_3 lies in β_2 therefore A_2 lies in α_3 and so, by Theorem 1, A_4 must lie in α_3.

Similarly in case (6) we may prove that A_3 must lie in α_4.

Consider next case (3).

We have A_4 in β_2 and therefore A_2 in α_4.

Also we have A_2 in α_1, and so A_1 in β_2.

Further we have A_4 in β_1, and so A_1 in α_4.

Thus A_4 and A_2 determine an optical line which contains A_1.

But A_3 is in α_2, and being distinct from A_2 it must be *after* A_2 an element of the optical line determined by A_4 and A_2.

Also since A_3 is in both α_1 and α_2 it follows that both A_1 and A_2 lie in β_3.

But by Post. IX (b) if A_3 were not in the optical line determined by A_4 and A_2 there would be *one single element* which would be an element both of the optical line and the sub-set β_3.

Thus since there are at least two distinct elements A_1 and A_2 common to the optical line and the sub-set β_3 it follows that A_3 must be an element of the optical line A_4A_2. Further, since A_3 lies in α_2 it must, by the definition of the optical line, lie also in α_4.

We may in a similar manner show in case (7) that A_4 must lie in α_3.

We are thus left with only case (5) to prove.

Now since A_2 is an element distinct from A_1 and lying in α_1, therefore, by Post. VIII (a), there is at least one other element distinct from A_2 which is a member both of α_1 and of α_2.

Call such an element A_5. Then A_2 is *before* A_5.

But A_3 is distinct from A_2 and lies in β_2 and so A_3 is *before* A_2. Thus A_3 is *before* A_5.

Also A_3 is distinct from A_1 and lies in α_1 and so A_3 is *after* A_1.

Thus, by Theorem 1, A_3 must be an element of the sub-set β_5.

Similarly A_4 must be an element of the sub-set β_5.

Also both A_3 and A_4 are elements of β_2 and so by Theorem 8 (b) one of the two elements A_3 and A_4 is in the β sub-set of the other, and therefore by Post. VII (b) one is in the α sub-set of the other.

Thus the theorem is true in all cases.

It follows directly from this theorem that of any two distinct elements in an optical line one is after *the other.*

THEOREM 10

Any two elements of an optical line determine that optical line.

Let A_1 be any element and A_2 any other element in α_1, then the optical line $A_1 A_2$ is defined as the aggregate of all elements which lie either

(1) both in α_1 and α_2,

or (2) both in α_1 and β_2,

or (3) both in β_1 and β_2.

Suppose A_3 and A_4 to be any pair of elements in the optical line $A_1 A_2$; then by Theorem 9 one of the pair A_3, A_4 is in the α sub-set of the other.

We may suppose without loss of generality that it is A_4 which is in the sub-set α_3.

Consider now any element A_5 of the optical line $A_1 A_2$ such that A_5 is distinct from A_3 and A_4.

Then by Theorem 9 there are the following possibilities:

A_4 in α_5 and also A_3 in α_5 ......(1),

A_4 in α_5 and also A_5 in α_3 ......(2),

A_5 in α_4 and also A_5 in α_3 ......(3),

A_5 in α_4 and also A_3 in α_5 ......(4).

Case (4) must however be excluded, for since A_3, A_4 and A_5 are supposed distinct we should have A_5 *after* A_4 and A_3 *after* A_5 and therefore, by Post. III, A_3 *after* A_4.

We however supposed A_4 to be *after* A_3 and by Post. I we cannot have also A_3 *after* A_4. Thus case (4) is impossible.

The three permissible cases may be expressed thus:

A_5 both in β_3 and β_4 ......(1),

A_5 both in α_3 and β_4 ......(2),

A_5 both in α_3 and α_4 ......(3).

Thus in all cases A_5 lies in the optical line defined by A_3 and A_4.

Similarly it may be shown that every element in the optical line defined by A_3 and A_4 lies in the optical line defined by A_1 and A_2.

Thus the optical lines $A_1 A_2$ and $A_3 A_4$ are identical.

THEOREM 11

If A_3 and A_4 be any two elements of an optical line A_1A_2 there is at least one element of the optical line which is after *the one and* before *the other.*

Since A_3 and A_4 are both elements of the same optical line the one must be in the α sub-set of the other by Theorem 9.

We shall suppose that A_4 lies in α_3.

Then since A_3 and A_4 are distinct, A_4 will be *after* A_3, and so by Theorem 5 there is at least one other distinct element which is a member both of α_3 and of β_4.

Call such an element A_5.

Then A_5 is in the optical line A_3A_4, and therefore by Theorem 10 in the optical line A_1A_2.

Further since A_5 is distinct from A_3 and A_4 it must be *after* A_3 and *before* A_4.

From the preceding results it follows that an optical line contains an infinite number of elements.

THEOREM 12

If an element A_1 be before *an element of an optical line a, and be also* after *an element of a, then A_1 must be itself an element of the optical line a.*

Suppose that A_1 is *before* the element A_2 of a and also *after* the element A_3 of a.

Then by Post. I A_3 cannot be identical with A_2, and by Theorem 9 one of the elements A_2 and A_3 must be in the α sub-set of the other.

Since A_1 is *after* A_3 and A_2 is *after* A_1 it follows that A_2 is *after* A_3 and so it must be A_2 which is in the α sub-set of A_3.

But, by Theorem 1, it follows that A_1 must lie in α_3 and also in β_2, and accordingly A_1 lies in the optical line A_3A_2.

Thus since, by Theorem 10, any two elements of an optical line determine that optical line, it follows that A_1 lies in the optical line a.

THEOREM 13

(*a*) *If A_1 be any element and A_2 be any other element in α_1 and if A_3 be any element in α_1 which is either* before *or* after *A_2, then A_3 lies in the optical line A_1A_2.*

(1) Suppose A_3 is *before* A_2.

Then since A_3 lies in α_1 it must be either identical with A_1, in which case it lies in the optical line A_1A_2; or else A_3 is *after* A_1, in which case by Theorem 1 A_3 must lie both in α_1 and β_2 and therefore must lie in the optical line A_1A_2.

(2) Suppose A_3 is *after* A_2.

Then A_3 lies in α_1 and A_2 is *after* A_1 and *before* A_3 and therefore, by Theorem 1, A_2 must lie both in α_1 and β_3.

But if A_2 lies in β_3, it follows by Post. VII (*b*) that A_3 lies in α_2.

Thus A_3 lies both in α_1 and α_2 and therefore lies in the optical line A_1A_2.

(*b*) *If A_1 be any element and A_2 be any other element in β_1 and if A_3 be any element in β_1 which is either* after *or* before *A_2, then A_3 lies in the optical line A_1A_2.*

Theorem 14

Three distinct elements cannot lie in pairs in three distinct optical lines.

Let A_1, A_2 and A_3 be three distinct elements and let A_1 and A_2 lie in one optical line.

We may suppose that it is A_2 which lies in α_1.

If then A_1 and A_3 lie in an optical line we may suppose either that A_3 lies in α_1 or in β_1.

First suppose A_3 lies in α_1.

Then if A_3 and A_2 lie in an optical line one of them must be *after* the other and so by Theorem 13 (*a*) A_3 must lie in the optical line A_1A_2.

Next suppose that A_3 lies in β_1.

Then if A_3 and A_2 lie in one optical line, A_1 is *before* A_2 one element of it and *after* A_3 another element of it and so by Theorem 12 A_1 must lie in the optical line A_3A_2. Thus the optical lines are not distinct and so the theorem is proved.

Remarks

If a and b be two distinct optical lines having an element E in common and if O be any element of a which is *before* E while D and F are elements of b which are respectively *before* and *after* E; then, E being *after* O, we must have F *after* O, but, by the last theorem, F and O cannot lie in an optical line.

Further, D cannot be *before* O, for then we should have O *after* one element of the optical line b and *before* another element of it and yet not lie in the optical line which, by Theorem 12, is impossible.

Also D cannot be *after* O, for then we should have D *after* one element of the optical line a and *before* another element of it and yet not lie in the optical line, which again is impossible.

Thus D is neither *before* nor *after* O.

Again, if b' be an optical line distinct from a but having an element E' in common with a and such that the element O of a is *after* E', while D' and F' are elements of b' which are respectively *after* and *before* E'; we may show in a similar way that F' is *before* O, but is not in an optical line with it; while D' is neither *after* nor *before* O.

THEOREM 15

(*a*) *If A_1 be any element and A_2 be any other element in α_1 and A_3 be any element in α_1 distinct from A_2 which is neither* before *nor* after *A_2, then A_3 is neither* before *nor* after *any element of the optical line A_1A_2 which is* after *A_1.*

The element A_3 cannot lie in the optical line A_1A_2, for then since it is distinct from A_2 it would be either *before* or *after* it, contrary to hypothesis.

Now any element of the optical line A_1A_2 which is *after* A_1 must lie in α_1.

Let A_4 be any such element.

Then if A_3 were either *before* or *after* A_4 it would by Theorem 13 lie in the optical line A_1A_4, which by Theorem 10 is identical with the optical line A_1A_2, and this we have shown to be impossible.

Thus A_3 cannot be either *before* or *after* any element of the optical line A_1A_2 which is *after* A_1.

(*b*) *If A_1 be any element and A_2 be any other element in β_1 and A_3 be any element in β_1 distinct from A_2 which is neither* after *nor* before *A_2, then A_3 is neither* after *nor* before *any element of the optical line A_2A_1 which is* before *A_1.*

POSTULATE X. (*a*) **If a be an optical line and if A be any element not in the optical line but before some element of it, there is one single optical line containing A and such that each element of it is before an element of a.**

(*b*) **If a be an optical line and if A be any element not in the optical line but after some element of it, there is one single optical line containing A and such that each element of it is after an element of a.**

THEOREM 16

(*a*) *If each element of one optical line be* before *an element of another optical line the two optical lines cannot have an element in common.*

Let a and b be two distinct optical lines such that each element of b is *before* an element of a.

Suppose, if possible, that a and b have an element A_1 in common.

Let A_2 be any element of b which is *after* and therefore distinct from A_1.

Then, by hypothesis, A_2 is *before* some element (say A_3) of a.

Thus we should have A_2 *after* one element A_1 and *before* another element A_3 of the optical line a and therefore, by Theorem 12, it would follow that A_2 must be an element of the optical line a.

Thus a and b would have two elements in common and so could not be distinct optical lines, contrary to hypothesis.

Thus the supposition that a and b have an element in common leads to a contradiction and is therefore impossible.

(*b*) *If each element of one optical line be* after *an element of another optical line the two optical lines cannot have an element in common.*

Theorem 17

(*a*) *If each element of an optical line a be* before *an element of another optical line b, then through each element of a there is* one single optical line *which contains also an element of b.*

By Theorem 16 an element of a cannot also be an element of b.

Suppose then that A_1 be any element of a.

Then A_1 is not an element of b, but is *before* an element of b and therefore by Post. IX (a) there is *one single element*, say A_2, which is an element both of the optical line b and the sub-set α_1. Since A_2 cannot be identical with A_1 it follows that A_1 and A_2 determine an optical line which contains an element of a and also an element of b.

Further, there cannot be more than one optical line through A_1 which contains also an element of b; for such an element of b must, by Theorem 9, lie either in α_1 or β_1.

But by Post. IX (a) there is only *one single element* common to b and the sub-set α_1, and so if such an element of b existed it would have to lie in β_1.

Call such a hypothetical element A_3.

Then since A_3 is supposed to lie in β_1, we should have A_1 in α_3.

But A_2 lies in α_1 and so A_1 lies in β_2, and thus if such an element as A_3 existed, A_1 would lie in the optical line A_3A_2: that is, in the optical line b, which is impossible, and so there cannot be any such element as A_3.

Thus there is only one single optical line through A_1 which contains also an element of b.

(*b*) *If each element of an optical line a be* after *an element of another optical line b, then through each element of a there is* one single optical line *which contains also an element of b.*

Definition. If two distinct optical lines have an element in common they will be said to *intersect* one another in that element.

If A_1 and A_2 be two distinct elements one of which is neither *before* nor *after* the other, then we know by Post. VI that there is at least one element, say X, which is *after* both A_1 and A_2, but is not *after* any other element which is *after* both A_1 and A_2.

From the definition of α sub-sets it follows that X lies both in α_1 and α_2, so that there is *at least one element* which is a member both of α_1 and α_2. Similarly there is *at least one element* which is a member both of β_1 and β_2.

These remarks prepare the way for Post. XI (*a*) and (*b*).

POSTULATE XI. (*a*) **If A_1 and A_2 be two distinct elements one of which is neither before nor after the other and X be an element which is a member both of α_1 and α_2, then there is at least one other element distinct from X which is a member both of α_1 and α_2.**

(*b*) **If A_1 and A_2 be two distinct elements one of which is neither after nor before the other and X be an element which is a member both of β_1 and β_2, then there is at least one other element distinct from X which is a member both of β_1 and β_2.**

The above is the first of our postulates which requires more than two dimensions for its representation.

It is to be noted that it may easily be combined with Post. VI as follows:

(*a*) *If A and B be two distinct elements one of which is neither* before *nor* after *the other, there are at least two distinct elements either of which is* after *both A and B but is not* after *any other element which is* after *both A and B.*

(*b*) *If A and B be two distinct elements one of which is neither* after *nor* before *the other, there are at least two distinct elements either of which is* before *both A and B but is not* before *any other element which is* before *both A and B.*

THEOREM 18

(*a*) *If* A_1 *and* A_2 *be any two distinct elements one of which is neither* before *nor* after *the other, and if* A_3 *and* A_4 *be distinct elements which lie both in* α_1 *and* α_2, *then one of these latter two elements is neither* before *nor* after *the other.*

By the definition of α sub-sets A_3 is *after* both A_1 and A_2 but is not *after* any other element which is *after* both A_1 and A_2.

Similarly A_4 is *after* both A_1 and A_2 but is not *after* any other element which is *after* both A_1 and A_2.

Thus A_3 is not *after* A_4, and A_4 is not *after* A_3.

Thus A_3 is neither *before* nor *after* A_4.

(*b*) *If* A_1 *and* A_2 *be any two distinct elements, one of which is neither* after *nor* before *the other, and if* A_3 *and* A_4 *be distinct elements which lie both in* β_1 *and* β_2, *then one of these latter two elements is neither* after *nor* before *the other.*

THEOREM 19

(*a*) *If* A_1 *be any element and* A_2 *and* A_3 *be two other distinct elements of* α_1, *one of which is neither* before *nor* after *the other, there is at least one other distinct element in* α_1 *which is neither* before *nor* after A_2 *and neither* before *nor* after A_3.

Since A_2 is a member of α_1, therefore A_1 is a member of β_2.

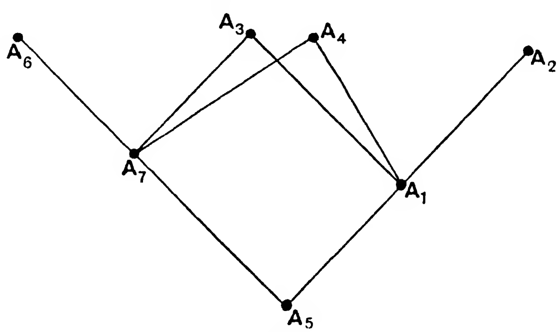


Fig. 5.

Thus by Post. VIII (*b*) there is at least one other element distinct from A_1 which is a member both of β_2 and of β_1.

Call such an element A_5.

Then A_1 and A_2 are both members of α_5.

Thus by Theorem 2 (*a*) there is at least one other element in α_5 distinct from A_1 which is neither *before* nor *after* A_1.

Call such an element A_6.

Now A_3 cannot lie in α_5 for then, as it is an element of α_1, it would lie in the optical line A_5A_1 along with A_2 and so A_2 and A_3 would either be identical or else A_2 would be either *before* or *after* A_3, contrary to hypothesis.

Now A_3 is *after* A_1 and A_1 is *after* A_5 and so by Post. III A_3 is *after* A_5, and since A_3 is not an element of α_5 it cannot lie in the optical line A_5A_6.

Thus by Post. IX (*b*) there is *one single element* (say A_7) which is an element both of the optical line A_5A_6 and the sub-set β_3.

Now A_5 cannot be *after* A_7, for A_3 lies in α_7 and so, by Theorem 1, A_5 would require to lie in β_3, which it cannot do since A_3 is not an element of α_5.

Also A_5 cannot coincide with A_7 for then it would be in β_3.

Thus A_7 must be *after* A_5, and so by Theorem 15 A_1 is neither *before* nor *after* A_7.

Now A_3 lies both in α_1 and in α_7, and so by Post. XI (*a*) there is at least one other distinct element, say A_4, which lies both in α_1 and in α_7.

Then by Theorem 18 A_4 is neither *before* nor *after* A_3.

Further, A_4 cannot be either *before* or *after* A_2, for since A_2 and A_4 are both members of α_1 it would follow by Theorem 13 that A_4 must lie in the optical line A_1A_2.

This would also be the case if A_4 coincided with A_2.

But then (since A_4 is *after* A_1 and therefore *after* A_5) we should have A_4 in α_5 and A_1 and A_7 both in α_5 and β_4, and thus A_1 and A_7 would lie in one optical line.

Thus A_1 and A_7 would either coincide or else the one would be *after* the other, which is impossible.

Thus A_4 is neither *before* nor *after* A_2 and is neither *before* nor *after* A_3 and is distinct from either.

(*b*) *If A_1 be any element and A_2 and A_3 be two other distinct elements of β_1, one of which is neither* after *nor* before *the other, there is at least one other distinct element in β_1 which is neither* after *nor* before *A_2 and neither* after *nor* before *A_3.*

THEOREM 20

If A_1 be any element there are at least three distinct optical lines containing A_1.

Let A_2 be any element in α_1 distinct from A_1.

Then by Theorem 2 (*a*) there is at least one other element in α_1 distinct from A_2 which is neither *before* nor *after* A_2.

Call such an element A_3.

Further by Theorem 19 there is at least one other distinct element in α_1 which is neither *before* nor *after* A_2 and neither *before* nor *after* A_3. Call such an element A_4.

Then A_1 and A_2 determine one optical line; A_1 and A_3 determine a second optical line; A_1 and A_4 determine a third optical line.

These are all distinct and all contain A_1.

If a be an optical line and if A be any element not in the optical line but *before* some element of it we have by Post. X (a) one single optical line containing A and such that each element of it is *before* an element of A.

Further, we have seen in Theorem 17 that there is one single optical line containing A and also intersecting a.

Also by Theorem 20 there are at least three optical lines containing A and so there must be at least one optical line containing A in addition to the two particular ones which we have already mentioned.

Similarly if a be an optical line and if A be any element not in the optical line but *after* some element of it, there is one single optical line containing A and such that each element of it is *after* an element of a and there is one single optical line containing A and intersecting a.

In addition to these two particular optical lines Theorem 20 shows that there is at least one other optical line containing A.

These considerations prepare the way for Post. XII (a) and (b).

POSTULATE XII. (a) **If a be an optical line and if A be any element not in the optical line but before some element of it, then each optical line through A, except the one which intersects a and the one of which each element is before an element of a, has one single element which is neither before nor after any element of a.**

(b) **If a be an optical line and if A be any element not in the optical line but after some element of it, then each optical line through A, except the one which intersects a and the one of which each element is after an element of a, has one single element which is neither after nor before any element of a.**

THEOREM 21

(a) *If each element of an optical line a be* after *an element of a distinct optical line b, then each element of b is* before *an element of a.*

Let A_1 be any element of a; then since A_1 is not in b but *after* an element of b, there is one single element (say A_2) common to the optical line b and the sub-set β_1 (Post. IX (b)).

Then A_2 is not an element of a but is *before* the element A_1 of a and so by Post. X (a) there is *one single optical line* (say c) containing A_2 and such that each element of it is *before* an element of a.

Now b cannot be identical with the optical line A_2A_1, for then a and b would have the element A_1 in common, which is impossible by Theorem 16 (b).

Suppose now, if possible, that b is not identical with c; then by Post. XII (a) there will be *one single element* in b (say A_3) which will be neither *before* nor *after* any element of a.

Consider an element A_4 in b and *after* A_3.

Since there can only be one element in b which is neither *before* nor *after* any element of a, it would follow that A_4 must be either *before* or *after* some element of a.

Since A_3 is *before* A_4 it would follow, if A_4 were *before* an element of a, that A_3 was also *before* an element of a, contrary to hypothesis.

We should therefore require A_4 to be *after* some element (say A_5) of a: so that A_5 would be *before* A_4: an element of b.

But by hypothesis A_5 is *after* some element of b and so, by Theorem 12, A_5 would require to lie in b.

Thus a and b would have an element in common, which is impossible by Theorem 16 (b).

Thus the supposition that b is distinct from c leads to a contradiction and therefore is not true.

Thus b must be identical with c and so each element of b is *before* an element of a.

(b) *If each element of an optical line a be* before *an element of a distinct optical line b, then each element of b is* after *an element of a.*

Theorem 22

If a be an optical line and if A_1 be any element which is neither before *nor* after *any element of a, there is* one single optical line *containing A_1 and such that no element of it is either* before *or* after *any element of a.*

Let A_2 be any selected element of a; then A_1 is neither *before* nor *after* A_2, and so by Post. VI (b) an element exists which is a member both of β_1 and of β_2.

Call such an element A_3.

Now A_3 is *before* A_2, an element of a, and does not lie in a and therefore by Post. X (a) there is *one single optical line* (say c) containing A_3 and such that each element of c is *before* an element of a.

Further, A_1 is *after* A_3, but is not *before* any element of a, and so does not lie in c.

Thus by Post. X (b) there is *one single optical line* (say b) containing A_1 and such that each element of b is *after* an element of c.

Consider now any element A_4 other than A_1 in the optical line b; then A_4 cannot be an element of a since otherwise A_1 would be either *before* or *after* an element of a, contrary to hypothesis.

Suppose now if possible that A_4 is *after* some element of a.

Then by Post. X (b) there is *one single optical line* (say d) containing A_4 and such that each element of d is after an element of a.

But since each element of a is *after* an element of c therefore by Post. III each element of d is *after* an element of c.

But by Post. X (b) there is only *one single optical line* containing A_4 which has this property and the optical line b is such a one.

Thus the optical line d must be identical with the optical line b.

Thus each element of b would be *after* an element of a, contrary to the hypothesis that A_1 was neither *before* nor *after* any element of a.

Thus A_4 is not *after* any element of a.

Next suppose if possible that A_4 is *before* some element (say A_5) of a.

Then A_5 is not an element of b, but is *after* an element of b, and so by Post. X (b) there is *one single optical line* (say e) containing A_5 and such that each element of e is *after* an element of b.

But each element of b is *after* an element of c and so by Post. III each element of e is *after* an element of c.

There is however by Post. X (b) one *one single optical line* containing A_5 and having this property, and a is such an optical line.

Thus e must be identical with a, and so each element of a must be *after* an element of b.

But if this were so then by Theorem 21 (a) each element of b must be *before* an element of a, contrary to the hypothesis that A_1 is neither *before* nor *after* any element of a.

Thus A_4 is not *before* any element of a, and so no element of b is either *before* or *after* any element of a.

We have thus shown that there is one optical line containing A_1 and having this property.

We have now to show that there is only one.

Consider any optical line containing A_1 other than the optical lines b and A_3A_1.

Call such an optical line f.

Then by Post. XII (b) there is *one single element* in f (say A_6) such that A_6 is neither *before* nor *after* any element in c.

If then we take any element A_7 in f and *before* A_6, such an element cannot be *after* any element in c, for then A_6 being *after* A_7 would be *after* an element of c, contrary to hypothesis.

Also, since there is only one element having the property of A_6 and lying in f, therefore A_7 must be *before* some element of c.

But this element is *before* some element of a, and so A_7 is *before* some element of a.

Thus there is only one optical line containing A_1 and such that no element of it is either *before* or *after* any element of a.

Theorem 23

If a be an optical line and A_1 be any element which is neither before *nor* after *any element of a while b is the one single optical line containing A_1 and such that no element of it is either* before *or* after *any element of a, then every optical line through A_1, with the exception of b, is divided by A_1 into elements which are* before *an element of a and elements which are* after *an element of a.*

We proved in Theorem 22 that there is only one optical line through A_1 having the property of b.

Thus if we take any other optical line d through A_1 there must be at least one element of d which is either *before* or *after* some element of a.

Suppose first that there is an element A_3 which is *before* some element of a.

Then A_3 cannot be *after* A_1, for since there is an element of a *after* A_3 there would by Post. III be an element of a *after* A_1, contrary to hypothesis.

Thus A_3 must be *before* A_1.

Further, A_3 cannot be an element of a, for then A_1 would be *after* an element of a, contrary to hypothesis.

Thus A_3 is not an element of a but *before* an element of it, and so by Post. IX (a) there is *one single element* (say A_2) which is an element both of the optical line a and the sub-set α_3.

Further by Post. X (a) there is *one single optical line* (say c) containing A_3 and such that each element of it is *before* an element of a.

Then by Post. XII (a) since the optical line d contains A_3 and is not identical with either of the optical lines A_3A_2 or c it follows that there

is *one single element* of d which is neither *before* nor *after* any element of a.

But by hypothesis A_1 has this property and so every other element of d is either *before* or *after* an element of a.

However, as we have already seen, an element which is *after* A_1 in d cannot be *before* an element of a and so it must be *after* an element of a.

Similarly an element which is *before* A_1 in d cannot be *after* an element of a, for then A_1 would be *after* an element of a contrary to hypothesis, and so an element which is *before* A_1 in d must be *before* an element of a.

We arrive at the same conclusion if we start off by supposing the existence in d of an element A_3' which is *after* some element of a. Thus the theorem is proved.

THEOREM 24

(*a*) *If each element of each of two distinct optical lines a and b be* after *elements of a third optical line c, and if one element A_1 of the optical line b be* after *some element of the optical line a, then each element of b is* after *an element of a.*

Let b' be the *one single optical line* containing A_1 and such that each element of b' is *after* an element of a.

Then since each element of a is *after* an element of c therefore by Post. III each element of b' is *after* an element of c.

But by hypothesis each element of b is *after* an element of c, and b contains A_1 an element not in the optical line c but *after* some element of it.

Thus by Post. X (*b*), since there is only one single optical line containing A_1 and having this property, it follows that b' must be identical with b.

Thus each element of b is *after* an element of a.

(*b*) *If each element of each of two distinct optical lines a and b be* before *elements of a third optical line c, and if one element A_1 of the optical line b be* before *some element of the optical line a, then each element of b is* before *an element of a.*

THEOREM 25

(*a*) *If each element of each of two distinct optical lines a and b be* after *elements of a third optical line c, and if one element A_1 of the optical line b be neither* before *nor* after *any element of the optical line a, then no element of the optical line b is either* before *or* after *any element of the optical line a.*

Since A_1 is not an element of c but is *after* some element of it, therefore

by Post. IX (b), there is one single element (say A_3) which is common to the optical line c and the sub-set β_1.

Then since A_3 is not an element of a, but is *before* an element of a (Theorem 21 (a)), therefore by Post. IX (a) there is one single element (say A_2) which is common to the optical line a and the sub-set α_3.

The demonstration then follows as in Theorem 22.

(b) *If each element of each of two distinct optical lines a and b be* before *elements of a third optical line c, and if one element A_1 of the optical line b be neither* after *nor* before *any element of the optical line a, then no element of the optical line b is either* after *or* before *any element of the optical line a.*

This may be demonstrated in an analogous manner.

THEOREM 26

(a) *If an optical line a be such that no element of it is either* before *or* after *any element of the optical line c, and if another optical line b be such that each element of it is* before *an element of c, then each element of b is* before *an element of a.*

Since each element of b is *before* an element of c, it follows by Theorem 21 (b) that each element of c is *after* an element of b.

Let A_1 be any element of c.

Then since A_1 is not an element of b but is *after* an element of b, there is one single element common to the optical line b and the sub-set β_1 (Post. IX (b)).

Let A_2 be this element.

Then A_2 and A_1 determine an optical line.

But by Theorem 23 every optical line containing A_1 except c is divided by A_1 into elements which are *before* an element of a and elements which are *after* an element of a, and since A_2 is *before* A_1 and lies in the optical line A_1A_2, it follows that A_2 is also *before* an element of a and is not an element of a.

Thus by Post. IX (a) there is one single element (say A_3) common to the optical line a and the sub-set α_2.

Now A_3 is neither *before* nor *after* any element of c and therefore if an optical line a' be taken through A_3 such that each element of it is *after* an element of b, then by Theorem 25 (a) no element of a' is either *before* or *after* any element of c.

But by Theorem 22 there is only one optical line through A_3 having this property and a is such an optical line.

Thus a' is identical with a and so each element of a is *after* an element of b, and thus by Theorem 21 (a) each element of b is *before* an element of a.

(b) *If an optical line a be such that no element of it is either* after *or* before *any element of the optical line c, and if another optical line b be such that each element of it is* after *an element of c, then each element of b is* after *an element of a.*

THEOREM 27

(a) *If each element of an optical line a be* after *an element of a distinct optical line c, and each element of another optical line b be* before *an element of c, then each element of a is* after *an element of b.*

By Theorem 21 (b) each element of c is *after* an element of b, and since each element of a is *after* an element of c, therefore by Post. III each element of a is *after* an element of b.

(b) *If each element of an optical line a be* before *an element of a distinct optical line c, and each element of another optical line b be* after *an element of c, then each element of a is* before *an element of b.*

THEOREM 28

If two distinct optical lines a and b be such that no element of either of them is either before *or* after *any element of a third optical line c, then no element of a is either* before *or* after *any element of b.*

For suppose, if possible, that some element A_1 of a is *after* an element of b; then A_1 cannot lie in b and by Post. IX (b) there is one single element (say A_2) common to the optical line b and the sub-set β_1.

But by Theorem 23 every optical line through A_1 except a is divided by A_1 into elements which are *before* an element of c and elements which are *after* an element of c.

Thus since A_2 and A_1 determine an optical line through A_1, and since A_2 is *before* A_1, therefore A_2 must be *before* an element of c, contrary to the hypothesis that no element of b is either *before* or *after* any element of c.

Similarly if we suppose A_1 to be *before* an element of b we are led to a conclusion contrary to hypothesis.

Thus no element of a is either *before* or *after* any element of b.

Definitions. An optical line a will be said to be parallel to a second distinct optical line b when either:

(1) each element of a is *after* an element of b,

or (2) each element of a is *before* an element of b,

or (3) no element of a is either *before* or *after* any element of b.

In case (1) a will be said to be an *after-parallel* of b.

In case (2) a will be said to be a *before-parallel* of b.

In case (3) a will be said to be a *neutral-parallel* of b.

It follows from these definitions in conjunction with Theorem 21 that:

If an optical line a be parallel to an optical line b, then the optical line b is parallel to the optical line a.

Again, if a be any optical line and A be any element *not in the optical line*, A may be *before* an element of a, or may be *after* an element of a, but by Theorem 12 A cannot be *before* one element of a and *after* another element of a.

By Post. XII (a) and (b) it follows that A may be neither *before* nor *after* any element of a.

If A be *before* an element of a, then by Post. X (a), there is one single parallel to a containing A.

If A be *after* an element of a, then by Post. X (b), there is one single parallel to a containing A.

If A be neither *before* nor *after* any element of a, then by Theorem 22 there is one single parallel to a containing A.

Thus we can say in general:

If a be any optical line and A be any element which is not in the optical line, then there is one single optical line parallel to a and containing A.

Further, combining Theorems 24 (a), 24 (b), 25 (a), 25 (b), 26 (a), 26 (b), 27 (a), 27 (b), 28, we have the general result that:

If two distinct optical lines a and b are each parallel to a third optical line c, then the optical lines a and b are parallel one to another.

Definition. If a and b be any pair of distinct optical lines one of which is an after-parallel of the other, then the aggregate of all elements of all optical lines which intersect both a and b will be called an *inertia plane*.*

* In the first edition of this work the term *acceleration* plane was used instead of *inertia* plane. The change was made in order that the nomenclature might be more systematic.

Theorem 29

If a be an optical line there are an infinite number of distinct inertia planes which all contain a.

From Post. XII (a) and (b) it follows that there is at least one element, say A_1, which is neither *before* nor *after* any element of a.

If b be the one optical line through A_1 such that no element of it is either *before* or *after* any element of a, then by Theorem 23 every optical line through A_1 except b is divided by A_1 into elements which are *before* an element of a and elements which are *after* an element of a.

Let f be one particular optical line containing A_1 and distinct from b.

Let A_2 be any element in f other than A_1; then A_2 must be either *before* or *after* some element of a but is not itself an element of a.

Thus if an optical line c be taken through A_2 parallel to a, then c is either a before- or after-parallel of a and therefore along with a serves to define an inertia plane.

Let A_3 be another element of f distinct from A_2.

Then in order that A_3 should lie in the inertia plane defined by a and c it would have to lie in an optical line intersecting both a and c.

But since A_3 is distinct from A_2 and lies in the optical line f which also contains A_2 it must be either *before* or *after* A_2, and so by Post. IX (a) or Post. IX (b) there must be *one single element* which is an element both of the optical line c and the sub-set α_3 or β_3 as the case may be.

But the element A_2 is such an element and therefore the optical line f containing A_3 and A_2 is the only optical line which intersects c and contains A_3.

Thus in order that A_3 should lie in the inertia plane defined by a and c it would be necessary for f to intersect a and this we know it does not do since if it did A_1 would be either *before* or *after* an element of a, contrary to hypothesis.

If then A_3 be distinct from A_1 it is either *before* or *after* an element of a and so if we take the optical line through A_3 parallel to a, it will be either a before- or after-parallel of a.

Call such an optical line d.

Then d and a define another inertia plane which is distinct from that defined by c and a, since the latter does not contain A_3.

If any other element A_n in the optical line f be selected other than A_2 or A_3 and an optical line be taken through it parallel to a, then, provided A_n is distinct from A_1, the parallel to a through A_n will, along with a, define an inertia plane distinct from the others.

Thus each element of f except A_1 corresponds to a distinct inertia plane and the number of elements in f is infinite, while all the inertia planes contain a.

Thus there are an infinite number of distinct inertia planes all containing the optical line a.

From the last theorem it follows directly that it is permissible to speak of three or more inertia planes which have two elements in common.

This prepares the way for Post. XIII.

POSTULATE XIII. **If two distinct inertia planes have two elements in common, then any other inertia plane containing these two elements contains all elements common to the two first-mentioned inertia planes.**

THEOREM 30

If a and b be two distinct optical lines and if a be an after-parallel of b, then if c and d be two other distinct optical lines intersecting both a and b, one of these latter two optical lines is an after-parallel of the other.

Let the optical line c intersect b in A_1 and a in A_2 and let the other optical line d intersect b in A_3 and a in A_4.

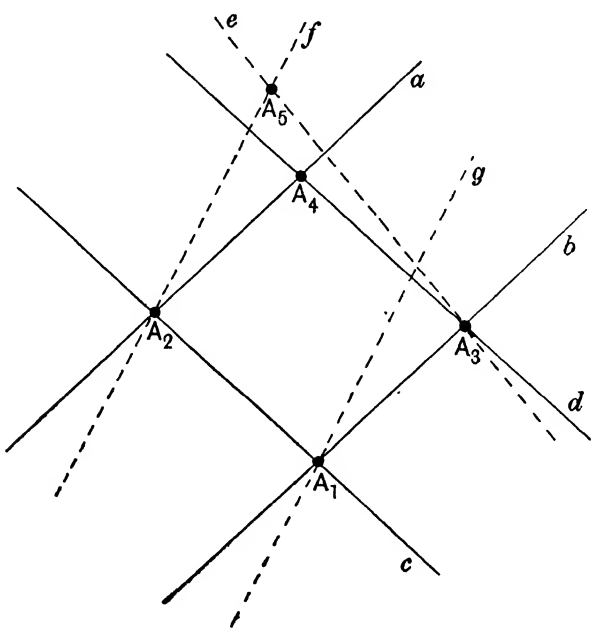


Fig. 6.

Then, by Theorem 17 (*a*), it is not possible for A_1 and A_3 to be coincident while A_2 and A_4 are distinct; while, by Theorem 17 (*b*), it is not possible for A_2 and A_4 to be coincident while A_1 and A_3 are distinct.

We may suppose without loss of generality that A_3 is *after* A_1.

Then since a is an after-parallel of b we must have A_4 *after* A_3 and therefore by Post. III A_4 is *after* A_1, or A_1 *before* A_4.

Further, since a is an after-parallel of b, and since A_1 and A_2 lie in the optical line c, we must have A_2 *after* A_1 and therefore A_2 must lie in α_1.

But now A_4 could not be *before* A_2, for then, by Theorem 1, A_4 would lie in α_1 and, since it is distinct from A_2, we should have two elements common to the optical line a and the sub-set α_1; which is impossible.

Thus since A_4 and A_2 both lie in the optical line a we must have A_4 *after* A_2 and so A_4 lies in α_2.

Now let e be the optical line through A_3 parallel to c; then e is an after-parallel of c since A_3 is *after* A_1.

Again there is one single optical line (say f) through A_2 intersecting e in some element, say A_5 which lies in α_2.

Now, since A_2 and A_3 are distinct elements both lying in α_1, and since A_2 does not lie in the optical line A_1A_3, it follows by Theorem 13 that A_2 is neither *before* nor *after* A_3 and therefore A_5 lies in α_3.

Suppose now, if possible, that A_5 is distinct from A_4; then by Theorem 18 (*a*) since A_4 and A_5 lie both in α_2 and α_3, the one is neither *before* nor *after* the other.

Thus A_5 could not lie either in a or d since then it would have to be either *before* or *after* A_4.

Neither can A_5 lie in b, for since A_2 is *after* A_1 and A_5 is *after* A_2, and A_1 is an element of b it would then follow by Theorem 12 that A_2 must lie in b, which is impossible.

Thus e is the only optical line through A_5 containing an element of b and if e also intersected a it would have to coincide with d, since d is the only optical line through A_3 which intersects a.

Thus if A_5 did not coincide with A_4 then A_5 could not lie in the inertia plane defined by a and b.

Thus the inertia plane defined by c and e would be distinct from the inertia plane defined by a and b.

Now let g be the optical line through A_1 parallel to f; then g is a before-parallel of f, since A_1 is *before* A_2.

Then g could not coincide with b for in that case we should have two optical lines a and f both through A_2 and both parallel to b, which is impossible.

Now A_3 lies in the optical line b which intersects g in A_1 and so if A_3 should lie in the inertia plane defined by f and g, then b would have to intersect f.

But the only optical line through A_1 intersecting f is c and so if A_3 should lie in the inertia plane defined by f and g, then b would have to coincide with c, which is impossible.

Thus A_3 would not lie in this inertia plane which therefore would be distinct from the inertia planes defined by a and b and by c and e, which both contain A_3.

But the inertia planes defined by a and b, by c and e, and by f and g all contain the two elements A_1 and A_2, while the two first-mentioned inertia planes also contain A_3, which would not be contained by the inertia plane defined by f and g.

This is contrary to Post. XIII and so the assumption that A_5 is distinct from A_4 must be abandoned.

Thus A_5 coincides with A_4 and therefore the optical line d coincides with the after-parallel of c through A_3.

This proves the theorem.

Theorem 31

If a, b, c, d, etc. be a set of parallel optical lines which all intersect one optical line l in elements A, B, C, D, etc., then through any element of one of the set of optical lines a, b, c, d, etc. other than the elements A, B, C, D, etc. there is one optical line which intersects each one of the set a, b, c, d, etc. and is parallel to l.

Since the elements A, B, C, D, etc. are elements of one optical line l, therefore of any two of these elements one is *after* the other by Theorem 9.

Thus of any two of the parallel optical lines a, b, c, d, etc. one is an after-parallel of the other.

If then one of these optical lines be selected (say b) and any element in it (say X) distinct from B there will be

one optical line through X intersecting a,
one optical line through X intersecting c,
one optical line through X intersecting d, etc.

But by Theorem 30 all these are parallel to l and since they all go through X they must all be identical.

Also for each element of b there is one such optical line and since any pair of such optical lines are parallel to l they are also parallel to one another.

This theorem shows that an inertia plane contains two sets of parallel optical lines which may be called the *generators* of the inertia plane.

Any generator of one set intersects every generator of the other set but does not intersect any one of its own set.

Also we see that through any element of an inertia plane there are two optical lines lying in the inertia plane and of these two one belongs to one set and the other to the other set.

Theorem 32

Through any element of an inertia plane there are only *two distinct optical lines which lie in the inertia plane.*

We have already seen that there are two optical lines which pass through any element of an inertia plane and lie in the inertia plane.

We have now to prove that there cannot be more than two.

Let A_1 be any particular element of an inertia plane and let a and b be the two generators of the inertia plane passing through A_1.

Suppose, if possible, that a third optical line c passes through A_1 and lies in the inertia plane.

Let A_2 be an element of c after A_1, then A_2 must lie in the inertia plane and so there would be two generators of the inertia plane passing through A_2 and parallel respectively to a and b.

The optical line parallel to a would meet b in some element, A_3 say, and the optical line parallel to b would meet a in A_4 say.

But if A_1, A_2 and A_3 were all distinct we should have three elements lying in pairs in three distinct optical lines, which is impossible by Theorem 14.

Similarly if A_1, A_2 and A_4 were all distinct.

Thus any optical line through A_1 and lying in the inertia plane must coincide either with a or b.

Theorem 33

If an inertia plane contain an optical line a and an element A_1 which does not lie in the optical line, then A_1 is either before *or* after *an element of a.*

There are two optical lines in the inertia plane which pass through A_1.

Of these two, one which we shall call b intersects a, while the other does not intersect it.

If b intersects a in an element A_2, then A_2 must be distinct from A_1 since A_1 does not lie in a.

But both A_1 and A_2 lie in the optical line b and so the one is *after* the other.

Thus A_1 is either *before* or *after* A_2: an element of the optical line a.

Theorem 34

If two elements be such that one is after *the other, but does not lie in an optical line with it, then there are an infinite number of inertia planes containing the two elements.*

Let A_1 and A_2 be the two elements and let A_2 be *after* A_1.

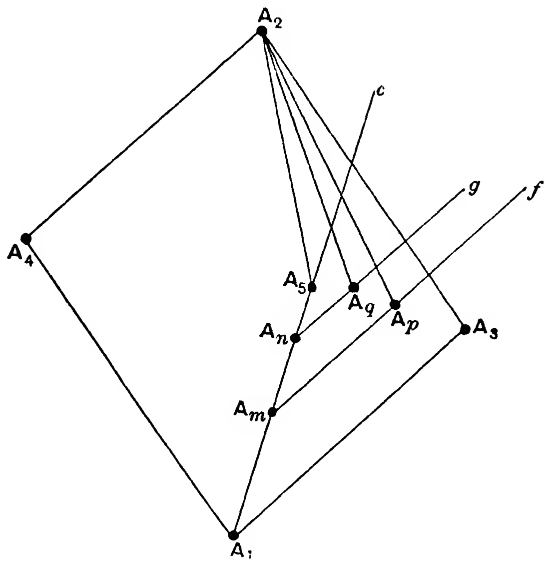


Fig. 7.

Then by Theorem 5 there is at least one other distinct element which is a member both of α_1 and of β_2. Call such an element A_3.

Then A_2 is in α_3 and so both A_1A_3 and A_3A_2 are optical lines.

But A_1 is not in the optical line A_3A_2 but is *before* A_3 an element of it and so we may take a before-parallel to A_3A_2 through A_1.

Then through A_2 there is one single optical line intersecting this before-parallel in some element, say A_4.

Then A_4A_2 will be an after-parallel of A_1A_3 by Theorem 30.

Now A_1A_3 and A_1A_4 are two distinct optical lines through A_1 and by Theorem 20 there are at least three distinct optical lines containing A_1 so that there must be at least one other. Let c be such an optical line.

Then A_2 is not in c but is *after* A_1 an element of c and so by Post. IX (b) there is *one single element* (say A_5) common to the optical line c and the sub-set β_2.

Then $A_1 A_5$ and $A_5 A_2$ are distinct optical lines and since A_2 is *after* A_5 we may take an after-parallel to $A_1 A_5$ through A_2, which together with $A_1 A_5$ will determine an inertia plane containing the given elements.

Let A_m and A_n be any two distinct elements of the optical line $A_1 A_5$ which are *after* A_1 and *before* A_5.

Then, A_m and A_n being elements which are *after* A_1 and not in the optical line $A_1 A_3$, we may take after-parallels to $A_1 A_3$ through A_m and A_n. Call these f and g respectively.

Then A_2 cannot be an element of f for then we should have the three elements A_m, A_5 and A_2 lying in pairs in three distinct optical lines, which is impossible by Theorem 14.

But A_5 is *after* A_m and A_2 is *after* A_5 and so by Post. III A_2 is *after* A_m an element of f.

Thus by Post. IX (b) there is *one single element* (say A_p) common to the optical line f and the sub-set β_2.

Similarly A_2 cannot be an element of g but is *after* A_n an element of g and so there is *one single element*, say A_q, common to the optical line g and the sub-set β_2.

Now A_m and A_n being both elements of the optical line $A_1 A_5$, the one must be *after* the other, and since f and g are both after-parallels of $A_1 A_3$ it follows by Theorem 24 that the one is an after-parallel of the other.

Thus f and g can have no element in common and so A_p and A_q must be distinct.

Further, A_p and A_q cannot both lie in the same optical line through A_2, for since f and g are both after-parallels of $A_1 A_3$ therefore by Theorem 31 this hypothetical optical line would also intersect $A_1 A_3$ and would therefore have to be identical with $A_3 A_2$. Thus the optical line $A_1 A_5$ or c would have to be parallel to $A_3 A_2$ and so be identical with $A_1 A_4$, contrary to hypothesis.

Thus the optical lines $A_p A_2$ and $A_q A_2$ must be distinct.

Further, either of them, say $A_p A_2$, must be distinct from $A_3 A_2$ for then $A_3 A_2$ would contain A_p an element of f, and since f is an after-parallel of $A_1 A_3$ therefore again $A_1 A_5$ would have to be identical with $A_1 A_4$, contrary to hypothesis.

Again, either of the optical lines $A_p A_2$ or $A_q A_2$ must be distinct from

A_5A_2, for, if we take A_pA_2, we should have the three elements A_m, A_p and A_5 lying in pairs in three distinct optical lines, which is impossible.

Similarly corresponding to each element of the optical line A_1A_5 which is *after* A_1 and *before* A_5 we may take an after-parallel to A_1A_3 which will have one element in common with the sub-set β_2 which determines a distinct optical line through A_2.

Since there are an infinite number of elements in the optical line A_1A_5 which are *after* A_1 and *before* A_5, it follows that there are an infinite number of optical lines through A_2 which are all distinct.

Since A_1 and A_2 are not in one optical line therefore A_1 cannot lie in any of these optical lines through A_2.

But A_1 is *before* A_2 and so by Post. X (a) a before-parallel to each of these optical lines may be taken through A_1 and the pair of parallel optical lines will determine an inertia plane containing A_1 and A_2.

Also since the number of optical lines through A_2 is infinite, and since by Theorem 32 only two optical lines pass through any element of an inertia plane and lie in the inertia plane, it follows that there are an infinite number of inertia planes containing the two elements A_1 and A_2.

Theorem 35

If two distinct elements be such that the one is neither before *nor* after *the other, then there are an infinite number of inertia planes containing the two elements.*

Let A_1 and A_2 be the two elements.

Then by Post. VI (a) and Post. XI (a) there are at least two other distinct elements which are members both of α_1 and α_2.

Let A_3 and A_5 be two such elements.

Then A_1A_3, A_1A_5, A_2A_3, A_2A_5 are distinct optical lines.

Let A_m and A_n be any two distinct elements of the optical line A_1A_5 which are *after* A_1 and *before* A_5.

Then A_m and A_n being elements which are *after* A_1 and not in the optical line A_1A_3, we may take after-parallels to A_1A_3 through A_m and A_n. Call these f and g respectively.

Then A_2 cannot be an element of f, for then we should have the three elements A_m, A_5 and A_2 lying in pairs in three distinct optical lines, which is impossible by Theorem 14.

But since f is an after-parallel of A_1A_3 it follows by Theorem 21 (a)

that A_1A_3 is a before-parallel of f and so A_3 is *before* some element of f or there is an element of f which is *after* A_3.

But A_3 is *after* A_2 and so by Post. III there is an element of f which is *after* A_2, or A_2 is *before* an element of f.

Thus by Post. IX (a) there is *one single element*, say A_p, which is an element both of the optical line f and the sub-set α_2.

Similarly A_2 cannot be an element of g but is *before* an element of g and so there is *one single element*, say A_q, common to the optical line g and the sub-set α_2.

Now A_m and A_n being both elements of the optical line A_1A_5, the one must be *after* the other, and since f and g are both after-parallels of A_1A_3 it follows by Theorem 24 that the one is an after-parallel of the other.

Thus f and g can have no element in common and so A_p and A_q must be distinct.

Further, A_p and A_q cannot both lie in the same optical line through A_2, for since f and g are both after-parallels of A_1A_3 it follows by Theorem 31 that this hypothetical optical line would also intersect A_1A_3 and would therefore have to be identical with A_2A_3.

Thus A_2A_3 would by Theorem 30 require to be either a before- or after-parallel of A_1A_5.

But A_3 is *after* A_1 and A_2 is *before* A_5 and so one element of A_2A_3 is *after* an element of A_1A_5 while another element of A_2A_3 is *before* an element of A_1A_5.

Thus A_2A_3 cannot be either a before or after-parallel of A_1A_5 and so A_p and A_q cannot both lie in the same optical line through A_2.

Thus the optical lines A_2A_p and A_2A_q must be distinct.

Further, either of them must be distinct from A_2A_3, for otherwise A_2A_3 would, again, require to be an after-parallel of A_1A_5, which we showed to be impossible.

Again, either of the optical lines A_2A_p, A_2A_q must be distinct from A_2A_5, for if we take for instance the case of A_2A_p, we should then have the three elements A_m, A_p and A_5 lying in pairs in three distinct optical lines, which is impossible.

Similarly corresponding to each element of the optical line A_1A_5 which is *after* A_1 and *before* A_5 we may take an after-parallel to A_1A_3 which will have one element in common with the sub-set α_2 which determines a distinct optical line through A_2.

Since there are an infinite number of elements in the optical line A_1A_5 which are *after* A_1 and *before* A_5, it follows that there

are an infinite number of optical lines through A_2 which are all distinct.

Since A_1 is neither *before* nor *after* A_2 and is distinct from it, therefore A_1 cannot lie in any of the optical lines through A_2.

Now there is only one element common to the optical line A_1A_5 and the sub-set α_2, namely the element A_5, and A_m cannot be *after* A_2 since otherwise, by Theorem 1, A_m would require to lie in α_2. But A_p is *after* A_2 and, since it lies in an optical line with A_m, A_p must be *after* A_m. But A_m is *after* A_1 and so A_p is *after* A_1. Similarly A_q is *after* A_1.

Thus A_1 is not an element of any of the optical lines through A_2 but is *before* elements of those which we have obtained, and so by Post. X (*a*) there is *one single optical line* containing A_1 and such that each element of it is *before* an element of any particular one of the optical lines through A_2 which we have obtained.

Each of these pairs of parallel optical lines determines an inertia plane containing A_1 and A_2 and, since the number of optical lines through A_2 is infinite, and since by Theorem 32 there are only two optical lines which pass through any element of an inertia plane and lie in the inertia plane, it follows that there are an infinite number of inertia planes containing the two elements A_1 and A_2.

Remarks

The last two theorems showed that an infinite number of inertia planes contain any pair of elements which *do not* lie in an optical line.

Further, Theorem 29 showed that an infinite number of inertia planes contain a given optical line and so contain any two elements which *do* lie in an optical line.

It is easy to show that if two or more distinct inertia planes contain an optical line there is no other element which they have in common which does not lie in the optical line.

Thus if we consider two inertia planes P and Q which both contain an optical line a, and suppose, if possible, that they have also an element A in common which does not lie in the optical line, then another optical line b through A must exist which is parallel to a.

The optical line b must lie in the inertia plane P and also in the inertia plane Q, and b must be either a before- or after-parallel of a, since A is either *before* or *after* an element of a (Theorem 33).

Thus a and b determine an inertia plane which would be identical both with P and Q, which could therefore not be distinct, contrary to hypothesis.

Thus if two or more inertia planes have an optical line in common they can have no other element outside the optical line in common.

We have also seen by Post. XIII that any inertia plane which contains two elements which are common to two distinct inertia planes, contains all elements common to them.

These remarks prepare the way for the following definitions and for Post. XIV.

Definitions. If two inertia planes contain two elements in common, then the aggregate of all elements common to the two inertia planes will be called a *general line.*

If two inertia planes contain two elements in common, of which one is *after* the other, but does not lie in the same optical line with it, then the aggregate of all elements common to the two inertia planes will be called an *inertia line.**

If two inertia planes contain two elements in common, of which one is neither *before* nor *after* the other, then the aggregate of all elements common to the two inertia planes will be called a *separation line.*†

POSTULATE XIV. (a) **If a be any inertia line and A_1 be any element of the set, then there is one single element common to the inertia line a and the sub-set α_1.**

(b) **If a be any inertia line and A_1 be any element of the set, then there is one single element common to the inertia line a and the sub-set β_1.**

THEOREM 36

An inertia line in any inertia plane has one single element in common with each optical line in the inertia plane.

Let a be the inertia line and let A_1 be an element in any optical line b in the inertia plane which we shall call P.

Then by Post. XIV (a) there is *one single element*, say A_2, common to the inertia line a and the sub-set α_1.

Also by Post. XIV (b) there is *one single element*, say A_3, common to the inertia line a and the sub-set β_1.

Now if A_1 lay in a, both A_2 and A_3 must coincide with A_1 since, if A_2 were distinct from A_1 we should have the two elements A_1 and A_2

* The name "inertia line" has been adopted because an inertia line represents the time path of an unaccelerated particle.

† The name "separation line" has been adopted because a single particle cannot occupy more than one element of a separation line, so that if particles P and Q occupy distinct elements of a separation line they must be *separate* particles.

in a which both lay in α_1, contrary to Post. XIV (a) which asserts that there is only *one single element* common to the inertia line a and the sub-set α_1.

Thus if A_1 lie in a, then A_2 must coincide with A_1.

Similarly if A_1 lie in a, then A_3 must coincide with A_1.

Suppose now that A_1 does not lie in a, then both A_2 and A_3 must be distinct from A_1.

Then we must have A_1 *after* A_3 and A_2 *after* A_1 and therefore A_2 *after* A_3; so that A_2 and A_3 must be distinct.

Also A_2 could not lie in α_3, for then we should have the two distinct elements A_2 and A_3 both common to the inertia line a and the sub-set α_3 contrary to Post. XIV (a). Thus A_2 and A_3 cannot lie in the same optical line.

But since A_2 is in α_1 and A_3 in β_1 it follows that A_1 and A_2 lie in an optical line through A_1, and also A_3 and A_1 lie in an optical line through A_1, and these optical lines are distinct and both lie in P.

Now by Theorem 32 there are only two distinct optical lines in the inertia plane which pass through A_1, and so one of them must be A_1A_2 and the other A_3A_1, and since b must be identical with one of these optical lines, it follows that a and b must have one single element in common.

Theorem 37

Of any two distinct elements of an inertia line one is after *the other.*

Let A_1 and A_2 be any two distinct elements of the inertia line a, and let b be one of the two optical lines in an inertia plane containing a which pass through A_1.

Now of the two optical lines in this inertia plane which pass through A_2, the one is parallel to b and the other intersects it in some element, say A_3.

Now A_1 and A_2 being distinct cannot both lie in α_3 by Post. XIV (a) and they cannot both lie in β_3 by Post. XIV (b).

Thus one of the two elements A_1 and A_2 must lie in α_3 and the other in β_3, and so one of them must be *after* A_3 and the other *before* A_3.

Thus by Post. III one of the two elements A_1 and A_2 must be *after* the other.

From the definition of a separation line it contains a pair of elements one of which is neither *before* nor *after* the other.

Thus it follows from the above theorem that *no inertia line can be a separation line and no separation line can be an inertia line.*

Theorem 38

If A_1 be any element in an inertia line a, there is at least one other element in the inertia line which is after A_1 *and also at least one other element in it which is* before A_1.

Let b be one of the two optical lines through A_1 in any inertia plane which contains a and let A_2 be any element in b which is *after* A_1.

Then by Post. XIV (a) there is one single element, say A_3, common to the inertia line a and the sub-set α_2.

Then A_3 cannot be identical with A_2 since then we should have two elements common to the inertia line a and the optical line b, contrary to Theorem 36.

Thus A_3 is *after* A_2 and A_2 is *after* A_1 and therefore A_3 is *after* A_1 and is an element of the inertia line a.

Similarly if we take any element A_4 in the optical line b and *before* A_1 there will by Post. XIV (b) be one single element, say A_5, common to the inertia line a and the sub-set β_4.

Then A_1 will be *after* A_4 and A_4 *after* A_5 and therefore A_1 *after* A_5.

Thus A_5 is *before* A_1 and is an element of the inertia line a.

Theorem 39

If A_1 and A_2 be any two distinct elements of an inertia line a, there is at least one other distinct element of a which is after *one of the two elements and* before *the other.*

By Theorem 37 one of the two elements A_1 and A_2 is *after* the other.

We shall suppose that A_2 is *after* A_1.

Let b and c be the two optical lines through A_1 in any inertia plane containing a.

Then the optical line through A_2 parallel to c will be an after-parallel and will intersect b in some element A_3 which must be *after* A_1.

Now let A_4 be any element in b which is *after* A_1 and *before* A_3 and consider the optical line through A_4 parallel to c.

This will be an after-parallel of c but a before-parallel of A_3A_2 and must intersect the inertia line a in some element, say A_5.

Then A_5 cannot be *before* any element of c and therefore is not *before* A_1.

Also A_5 cannot be *after* any element of A_3A_2 and therefore is not *after* A_2.

Thus by Theorem 37 A_5 must be *after* A_1 and *before* A_2 and lies in the inertia line a.

It follows from the above results that there are an infinite number of elements in any inertia line.

POSTULATE XV. **If two general lines, one of which is a separation line and the other is not, lie in the same inertia plane, then they have an element in common.**

Since there are an infinite number of optical lines in an inertia plane, and since only two of them pass through any given element, and since by Post. XV each of them has an element in common with any separation line lying in the inertia plane, it follows that there are an infinite number of elements in any separation line.

Further, since as we have remarked in connexion with Theorem 37 no inertia line can be a separation line, it follows that *no element of a separation line is either* before *or* after *another element of it.*

THEOREM 40

If A_1 and A_2 be two distinct elements one of which is neither before *nor* after *the other, and if a and b be the two optical lines through A_1 in an inertia plane containing the two elements, then A_2 is* before *an element of one of these optical lines and* after *an element of the other.*

By Theorem 33 A_2 must be either *before* or *after* an element of a and also must be either *before* or *after* an element of b; but A_2 cannot lie either in a or b since it is distinct from A_1 and is neither *before* nor *after* it.

Suppose first that A_2 is *before* an element of a.

Then one of the two optical lines through A_2 in the inertia plane will intersect a in some element, say A_3, while the other optical line through A_2 in the inertia plane will intersect b in some element, say A_4.

Then A_2 must be *before* A_3 since A_2 cannot either lie in a or be *after* any element of it.

But A_3 cannot either coincide with A_1 or be *before* A_1, for then we should have A_2 *before* A_1, contrary to hypothesis.

Thus A_3 must be *after* A_1.

But A_1 is an element of b and so the optical line A_2A_3 (which since it intersects a must be parallel to b) must be an after-parallel of b.

Thus A_2 must be *after* an element of b, and since A_2 must be either *before* or *after* A_4, it follows that A_2 is *after* A_4.

In a similar manner we may prove that if A_2 be *before* an element of b it must be *after* an element of a.

Also in an analogous manner we may show that if A_2 be *after* an element of b it must be *before* an element of a, and if A_2 be *after* an element of a it must be *before* an element of b.

Thus A_2 must be *before* an element of one of the optical lines a and b and *after* an element of the other.

Definition. An element in an inertia plane will be said to be *between* a pair of parallel optical lines in the inertia plane if it be *after* an element of the one optical line and *before* an element of the other and does not lie in either optical line.

Theorem 41

If A_1 and A_2 be any two distinct elements of a separation line, there is at least one other element of the separation line which lies between a pair of parallel optical lines through A_1 and A_2 respectively in an inertia plane containing the separation line.

Let a_1 and b_1 be the two optical lines passing through A_1 in any inertia plane containing the separation line.

Then, since A_2 is neither *before* nor *after* A_1, it follows that A_2 is *before* an element of one of the two optical lines a_1 and b_1 and is *after* an element of the other. (Theorem 40.)

Suppose that A_2 is *before* an element of a_1.

Then it is *after* an element of b_1.

Let a_2 and b_2 be the two optical lines through A_2 parallel respectively to a_1 and b_1.

Then a_2 and b_2 lie in the inertia plane and since A_2 is *before* an element of a_1 therefore a_2 is a before-parallel of a_1.

Similarly since A_2 is *after* an element of b_1 it follows that b_2 is an after-parallel of b_1.

Further, b_2 must intersect a_1 in some element, say A_3, which must be *after* A_2 since a_1 is an after-parallel of a_2.

Let A_4 be any element of b_2 which is *after* A_2 and *before* A_3 and consider the optical line through A_4 parallel to a_1.

We shall call this optical line a'.

Then since A_4 is *before* A_3 it follows that a' is a before-parallel of a_1 and since A_4 is *after* A_2 therefore a' is an after-parallel of a_2.

Also a' lies in the inertia plane.

Thus by Post. XV a' must have an element in common with the separation line A_1A_2.

Call this element A_5.

Then since a' is a before-parallel of a_1 therefore A_5 is *before* an

element of a_1 and since a' is an after-parallel of a_2 therefore A_5 is *after* an element of a_2.

Thus A_5 is between the parallel optical lines a_1 and a_2.

THEOREM 42

If A_1, A_2 and A_3 be three elements in a separation line and if A_3 lies between a pair of parallel optical lines through A_1 and A_2 in an inertia plane containing the separation line, then A_3 also lies between a second pair of parallel optical lines through A_1 and A_2 in the inertia plane.

Let a_1 and a_2 be a pair of parallel optical lines through A_1 and A_2 respectively in the inertia plane and suppose that A_3 lies between them.

We may without loss of generality suppose that A_3 is *after* an element of a_2 and *before* an element of a_1.

Let b_1 be the second optical line which passes through A_1 in the inertia plane and let b_2 be the second optical line which passes through A_2 in the inertia plane.

Then, since a_1 and a_2 are parallel, b_1 and b_2 are also parallel.

But since A_3 and A_1 lie in a separation line, A_3 is neither *before* nor *after* A_1, and since A_3 is *before* an element of a_1 therefore by Theorem 40 A_3 is *after* an element of b_1.

Similarly A_3 is neither *before* nor *after* A_2 and, since A_3 is *after* an element of a_2, therefore, by Theorem 40, A_3 is *before* an element of b_2.

Thus A_3 is between the parallel optical lines b_1 and b_2 passing through A_1 and A_2 respectively in the inertia plane.

Since there are only two optical lines in an inertia plane which pass through a given element of it, it follows directly from the above theorem that *if A_1, A_2 and A_3 be three elements in a separation line and if A_3 lies between a pair of parallel optical lines through A_1 and A_2 in an inertia plane containing the separation line, then A_2 does not lie between a pair of parallel optical lines through A_1 and A_3 in the inertia plane.*

Similarly A_1 does not lie between a pair of parallel optical lines through A_2 and A_3 in the inertia plane.

THEOREM 43

If A_1 and A_2 be any two elements of a separation line, there is at least one other element of the separation line such that A_2 lies between a pair of parallel optical lines through A_1 and that element in an inertia plane containing the separation line.

Using the notation employed in Theorem 41 let us take any element, say A_6, in the optical line b_2 and *before* A_2 and consider the optical line through A_6 parallel to a_2.

Call this optical line a''.

Then since A_6 is *before* A_2 therefore a'' is a before-parallel of a_2, and since a_2 is a before-parallel of a_1 therefore a'' is also a before-parallel of a_1.

Further, a'' lies in the inertia plane and so by Post. XV it has an element in common with the separation line.

Call this element A_7.

Then A_2 is *before* A_3 an element of a_1 and is *after* A_6 an element of a''.

Thus A_2 is between the parallel optical lines a_1 and a'' passing through A_1 and the element A_7 respectively and lying in the inertia plane.

THEOREM 44

Of any three distinct elements of a separation line in a given inertia plane there is one which lies between a pair of parallel optical lines through the other two and in the inertia plane.

Let A_1, A_2 and A_3 be any three distinct elements in the separation line.

Then, since there are two optical lines in an inertia plane passing through any element of it, let us select one of those passing through one of these elements, say A_1, and the parallel optical lines through A_2 and A_3.

Call these optical lines a_1, a_2 and a_3 respectively.

Then a_1, a_2 and a_3 all intersect any generator of the inertia plane belonging to the opposite set.

Let b be such a generator and suppose that a_1, a_2 and a_3 intersect b in the elements A_1', A_2' and A_3' respectively.

Then A_1', A_2' and A_3' being all elements of the optical line b, and being all distinct, it follows that of any two of them one must be *after* the other.

Thus remembering that Post. III must be satisfied it follows that either

$$\left.\begin{array}{ll} & A_2' \text{ is } \textit{after } A_1' \text{ and } A_3' \textit{ after } A_2' \quad (1), \\ \text{or} & A_2' \text{ is } \textit{after } A_3' \text{ and } A_1' \textit{ after } A_2' \quad (2), \end{array}\right\}$$

$$\left.\begin{array}{ll} \text{or} & A_3' \text{ is } \textit{after } A_1' \text{ and } A_2' \textit{ after } A_3' \quad (3), \\ \text{or} & A_3' \text{ is } \textit{after } A_2' \text{ and } A_1' \textit{ after } A_3' \quad (4), \end{array}\right\}$$

$$\left.\begin{array}{ll} \text{or} & A_1' \text{ is } \textit{after } A_2' \text{ and } A_3' \textit{ after } A_1' \quad (5), \\ \text{or} & A_1' \text{ is } \textit{after } A_3' \text{ and } A_2' \textit{ after } A_1' \quad (6). \end{array}\right\}$$

In case (1) a_2 is an after-parallel of a_1 and a before-parallel of a_3 and so each element of a_2 is between the parallel optical lines a_1 and a_3.

Thus A_2 is between a pair of parallel optical lines through A_1 and A_3 in the inertia plane.

Similarly in case (2) a_2 is an after-parallel of a_3 and a before-parallel of a_1 and therefore again A_2 is between a pair of parallel optical lines through A_1 and A_3 in the inertia plane.

In a similar manner in cases (3) and (4) A_3 is between a pair of parallel optical lines through A_1 and A_2 in the inertia plane; while in cases (5) and (6) A_1 is between a pair of parallel optical lines through A_2 and A_3 in the inertia plane.

Thus in all cases one of the three elements is between a pair of parallel optical lines through the other two and in the inertia plane.

Theorem 45

If A be an element of an optical line a and if B be an element which is neither before *nor* after *any element of a, then no element of the separation line AB, with the exception of A, is either* before *or* after *any element of a.*

Let C be any element of the separation line AB other than A, and let c be an optical line through C parallel to a.

Suppose, if possible, that C is either *before* or *after* some element of a.

Then c would be either a before- or after-parallel of a and accordingly c and a would be generators of an inertia plane which would contain the two elements A and C of the separation line AB and would therefore contain every element of AB.

Thus the element B would lie in an inertia plane containing the optical line a, and therefore, by Theorem 33, B would be either *before* or *after* an element of a, contrary to hypothesis.

Thus the assumption that any element of the separation line AB, other than A, is either *before* or *after* any element of a leads to a contradiction and therefore is not true and so no element of AB with the exception of A is either *before* or *after* any element of a.

Sets of three elements which determine inertia planes

Let A_1, A_2 and A_3 be three distinct elements which do not all lie in one general line, then A_1 and A_2 must lie in one general line, A_2 and A_3 in a second and A_3 and A_1 in a third.

These three general lines need not however lie in one inertia plane, although they do in certain cases.

In these latter cases the three elements determine the inertia plane containing them, since if they should lie in two distinct inertia planes they would lie in one general line, contrary to hypothesis.

It is important to have criteria by which we can say that a set of three elements does lie in one inertia plane.

Case I. Three elements A_1, A_2, A_3 lie in one inertia plane if A_1 and A_2 lie in an optical line while A_3 is an element not in the optical line but *before* some element of it, or *after* some element of it.

This is clearly true, since, if A_1 and A_2 lie in the optical line a, while A_3 does not lie in a but is *before* some element of it, then there is a before-parallel optical line, say b containing A_3, and so a and b are a pair of parallel generators of an inertia plane, containing A_1, A_2 and A_3 and which is determined by them.

Similarly if A_3 be *after* some element of a there is a definite after-parallel optical line b containing A_3, and the two optical lines a and b are a pair of parallel generators of an inertia plane containing A_1, A_2 and A_3 and which is determined by them.

Case II. Three elements A_1, A_2, A_3 lie in one inertia plane if A_1 and A_2 lie in an inertia line and A_3 be *any* element outside the inertia line.

This can also be readily seen to hold since if a denote the inertia line containing A_1 and A_2 then by Post. XIV (a) there is one single element, say A_4, common to the inertia line a and the sub-set α_3, and by Post. XIV (b) there is one single element, say A_5, common to the inertia line a and the sub-set β_3.

Thus A_3 and A_4 lie in one optical line while A_3 and A_5 lie in another optical line.

These two optical lines may be taken as generators of opposite sets of an inertia plane containing A_3, A_4 and A_5.

But since this inertia plane contains the two elements A_4 and A_5 of the inertia line a, it must contain every element of a and therefore contains A_1 and A_2.

Thus the three elements A_1, A_2 and A_3 lie in one inertia plane which is determined by them.

Case III. Three elements A_1, A_2, A_3 lie in one inertia plane if A_1 and A_2 lie in a separation line and if A_3 be an element not in the separation line but *before* at least two elements of it or *after* at least two elements of it.

In order to show this let a be the separation line containing A_1 and A_2 and suppose A_3 is *before* the elements A_4 and A_5 of a which are supposed distinct.

Then A_3 and A_4 must lie either in an optical line or an inertia line

since A_4 is *after* A_3, and similarly A_3 and A_5 must lie either in an optical line or an inertia line and the two general lines A_3A_4 and A_3A_5 are distinct.

If A_3A_4 and A_3A_5 be both optical lines, then they may be taken as generators of opposite sets of an inertia plane containing A_3, A_4 and A_5.

But this inertia plane, since it contains the two distinct elements A_4 and A_5 of the separation line a, must contain every element of it and so must contain A_1 and A_2.

Thus A_1, A_2 and A_3 lie in one inertia plane which is determined by them.

We shall suppose next that at least one of the general lines A_3A_4 and A_3A_5 is an inertia line.

Let us say that A_3A_4 is an inertia line.

Then by Case II the three elements A_3, A_4 and A_5 lie in one inertia plane which is determined by them.

But since this inertia plane contains the two elements A_4 and A_5 of the separation line a, therefore it contains every element of a and so must contain A_1 and A_2.

Thus A_1, A_2 and A_3 lie in one inertia plane which is determined by them.

The case when A_3 is *after* two distinct elements of a is quite analogous.

If A_1 and A_2 lie in an optical line a while A_3 is an element which is neither *before* nor *after* any element of a, then the three elements do not lie in one inertia plane, for by Theorem 45 no element of the general line A_1A_3 with the exception of A_1 is either *before* or *after* any element of a.

But if A_1, A_2 and A_3 should lie in an inertia plane there would be two optical lines through A_2 in the inertia plane and both of these would have an element in common with the separation line A_1A_3.

Thus there would be at least two elements of A_1A_3 which would be *before* or *after* A_2, contrary to Theorem 45.

Thus A_1, A_2 and A_3 do not lie in one inertia plane.

If A_1 and A_2 lie in a separation line a, while A_3 is *before* one *single* element of a or *after* one *single* element of a, then the three elements A_1, A_2, A_3 cannot lie in one inertia plane.

This is easily seen, for if we suppose that they do all lie in one inertia plane, there are two optical lines through A_3 in the inertia plane which have each an element in common with a.

If these elements be called A_4 and A_5 then, since a is a separation line, A_4 is neither *before* nor *after* A_5 and so A_4 and A_5 must be either both *before* or both *after* A_3, contrary to the hypothesis that there is only *one single* element of a which A_3 is *after* or *before*.

If A_1 and A_2 lie in a separation line a, while A_3 does not lie in a and is neither *before* nor *after* any element of a, it is also evident from the above considerations that the three elements A_1, A_2, A_3 cannot lie in one inertia plane.

We have not however as yet proved the possibility of this last case, but shall do so hereafter (Theorem 99). Till then no use will be made of it, and it is merely mentioned here for the sake of completeness.

Definition. If an inertia plane have its two sets of generators respectively parallel to the two sets of generators of another distinct inertia plane, then the two inertia planes will be said to be *parallel* to one another.

It is clear that if P be an inertia plane and A be any element outside it, then there is one single inertia plane containing A, and parallel to P; for there is one single optical line through A parallel to the one set of generators of P and one single optical line through A parallel to the other set of generators of P.

These are generators of opposite sets of an inertia plane containing A and determine that inertia plane, which is therefore unique.

It is further clear that two parallel inertia planes can have no element in common, for if the element A lies outside the inertia plane P and if a be an optical line passing through A and parallel to a generator of P, then a can have no element in common with P since otherwise it would require to lie entirely in P, contrary to the hypothesis that A is outside P.

Similarly any optical line b which intersects a and is parallel to a generator of P of the opposite set can have no element in common with P.

But if Q be the inertia plane passing through A and parallel to P, every element of Q must lie in an optical line such as b and so P and Q can have no element in common.

It is also clear from the definition that *two distinct inertia planes which are parallel to the same inertia plane are parallel to one another*; since distinct optical lines which are parallel to the same optical line are parallel to one another.

THEOREM 46

If an inertia plane P have one element in common with each of a pair of parallel inertia planes Q and R, then, if P have a second element in common with Q, it has also a second element in common with R.

If P and Q have two elements in common they must have a general line in common which we may call a.

Let B_1 be the element which by hypothesis P and R have in common.

Then if a be an inertia or separation line it follows by Theorem 36 and Post. XV that both the optical lines through B_1 in the inertia plane P have an element in common with a, while if a be an optical line one of the optical lines through B_1 in P has an element in common with a.

Thus in all cases at least one of the optical lines through B_1 in the inertia plane P has an element in common with a.

Let A_1 be such an element.

Suppose first that a is an optical line.

Then a is one of the generators of Q and since the inertia plane R is parallel to Q and since B_1 lies in R there will be one of the generators of R passing through B_1 and parallel to a.

Since A_1 and B_1 lie in an optical line and are distinct, the one must be *after* the other and so this parallel to a through B_1 must be either a before- or after-parallel of a.

Let us denote it by b.

Then a and b determine an inertia plane which contains three distinct elements of P which are not all in one general line and so this inertia plane must be identical with P.

Thus since it contains the optical line b it follows that P has a second element in common with R.

Suppose next that a is an inertia or separation line and let c be one of the generators of Q which pass through A_1.

Then since R is parallel to Q and since B_1 lies in R there will be one of the generators of R passing through B_1 and parallel to c.

Since A_1 and B_1 lie in an optical line and are distinct, the one must be *after* the other and so this parallel to c through B_1 must be a before- or after-parallel.

Let C be any element of c distinct from A_1 and let an optical line through C intersect the optical line through B_1 parallel to c in the element D.

Then by Theorem 30 the optical line CD must be a before- or after-parallel of the optical line A_1B_1.

Let the second optical line through C in the inertia plane Q meet a in the element A_2.

The element A_2 must exist since a is an inertia or separation line.

Since the optical line CA_2 must be a generator of Q of the opposite set to c, there must be an optical line through D in the inertia plane R which is parallel to CA_2 and is a generator of R of the opposite set to DB_1.

Since C and D lie in an optical line and are distinct, the one must be *after* the other and so this parallel to CA_2 through D must be a before- or after-parallel.

Let an optical line through A_2 intersect the optical line through D parallel to CA_2 in the element B_2.

Then by Theorem 30 the optical line A_2B_2 must be a before- or after-parallel of CD and CD is a before- or after-parallel of A_1B_1, and so if A_1B_1 and A_2B_2 be distinct they must be parallel to one another.

Now the optical lines CA_1 and CA_2 are distinct from the inertia or separation line a and are also distinct from one another.

Also the element C cannot lie in a since then CA_1 would have to be an inertia or separation line.

Thus the elements A_1 and A_2 are distinct and since they lie in an inertia or separation line they cannot lie in one optical line.

Thus A_2B_2 is distinct from A_1B_1 and is therefore parallel to it.

Also since the general line a and the optical line A_1B_1 lie in the inertia plane P and since the element A_2 does not lie in A_1B_1 it follows by Theorem 33 that A_2 is either *before* or *after* some element of A_1B_1.

Thus A_2B_2 must be either a before- or after-parallel of A_1B_1 and so the optical lines A_1B_1 and A_2B_2 lie in an inertia plane containing the general line a and the element B_1.

This inertia plane must therefore be identical with P and it contains the element B_2 in common with R where B_2 is distinct from B_1.

Thus the theorem holds in all cases.

REMARKS

It follows directly from this theorem that if two distinct inertia planes P and Q have a general line in common and, if further, P has one element in common with an inertia plane R which is parallel to Q, then P and R have a general line in common.

Further, since Q and R can have no element in common, it follows that these two general lines have no element in common.

Again if Q and R be two parallel inertia planes and if a be any general

line in Q, then there is at least one inertia plane containing a and another general line in R.

This may be shown in the following way:

Let A_1 be any element of a and let f be any inertia line in R.

Then by Post. XIV (a) there is one single element common to the inertia line f and the sub-set α_1. Let B be this element and let A_2 be any element of f which is *after* B.

Then A_2 is *after* A_1 but does not lie in α_1 and so A_1 and A_2 lie in an inertia line.

Thus A_2 and a lie in an inertia plane, say P, which by Theorem 46 must contain a second element in common with R.

Thus P contains a and another general line in R.

It is easy to see that there are really an infinite number of inertia planes which have this property of P.

We have seen that if two distinct optical lines intersect a pair of optical lines one of which is an after-parallel of the other, then of the two first-mentioned optical lines one is an after-parallel of the other (Theorem 30).

We have also seen that it is impossible for an optical line to intersect a pair of neutral-parallel optical lines.

Thus we may state the following definition:

Definition. If two distinct optical lines intersect a pair of optical lines one of which is an after-parallel of the other, then the four optical lines will be said to form an *optical parallelogram*.

It is evident that an optical parallelogram must lie in an inertia plane.

The elements of intersection will be spoken of as the *corners* of the optical parallelogram.

A pair of corners which lie in one optical line will be spoken of as *adjacent*.

A pair of corners which do not lie in one optical line will be spoken of as *opposite*.

A general line passing through a pair of opposite corners of an optical parallelogram will be spoken of as a *diagonal line* of the optical parallelogram.

We make a distinction between two optical parallelograms having a *diagonal line* in common and having a *diagonal* in common.

When we speak of two optical parallelograms having a *diagonal line* in common we shall mean that a pair of opposite corners of each of the optical parallelograms lie in the same general line.

When, on the other hand, we speak of two optical parallelograms having a *diagonal* in common, we mean that they have a pair of opposite corners in common.

It is obvious that an optical parallelogram has two diagonal lines and it is easy to see that *one of these must be an inertia line, and the other a separation line.*

For if we call the four optical lines a, b, c and d, and if a be an after-parallel of b while c is an after-parallel of d, then the intersection element of a and c must be *after* the intersection element of d and b so that these two intersection elements lie in an inertia line.

Further, if we denote the intersection element of a and c by A_1, that of a and d by A_2, that of c and b by A_3 and that of d and b by A_4 it follows by Theorem 13 (b) that if A_3 were either *before* or *after* A_2 then A_3 would have to lie in the optical line A_2A_1, or a contrary to hypothesis.

Thus A_3 is neither *before* nor *after* A_2 and so A_2 and A_3 lie in a separation line.

Definition. If a general line a have *one single element* in common with a general line b, then a will be said to *intersect* b.

Since a general line does not intersect itself and since we may have two optical parallelograms in the same inertia plane having a diagonal line in common, it is permissible to speak of two optical parallelograms in the same inertia plane whose diagonal lines of one kind or the other do not intersect.

This prepares the way for Post. XVI.

POSTULATE XVI. **If two optical parallelograms lie in the same inertia plane, then if their diagonal lines of one kind do not intersect, their diagonal lines of the other kind do not intersect.**

THEOREM 47

If a be any general line in an inertia plane P and A be any element of the inertia plane which is not in the general line, then there is one single general line *through A in the inertia plane which does not intersect a.*

Let Q be any other inertia plane distinct from P and containing the general line a, and let R be an inertia plane passing through A and parallel to Q.

Then by Theorem 46 P and R will have a general line in common which can have no element in common with a, and so there is at least

one general line through A in the inertia plane P which does not intersect a.

We must next show that there is only one such general line.

Consider first the case where a is an optical line.

Then of the two optical lines through A in the inertia plane P we know that one is parallel to a while the other intersects it.

Further by Theorem 36 any inertia line through A in the inertia plane P must intersect a.

Also by Post. XV any separation line through A in the inertia plane P must intersect a.

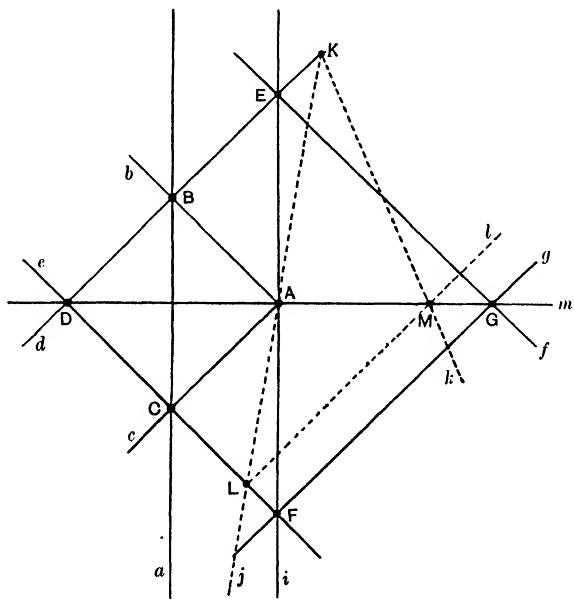


Fig. 8.

Thus if a be an optical line there is one single general line through A in the inertia plane P which does not intersect a.

Consider next the cases where a is an inertia or a separation line.

If a be an inertia line, then by Theorem 36 both the optical lines through A in the inertia plane P intersect a, while by Post. XV every separation line in P intersects a.

Thus when a is an inertia line any general line through A in the inertia plane P which does not intersect a can only be an inertia line.

Also from Post. XV it follows that when a is a separation line any general line through A in the inertia plane P which does not intersect a can only be a separation line.

With these provisos the demonstration of the unique character of the non-intersecting general line is similar in the two cases.

Suppose, if possible, that there are two general lines through A in the inertia plane, say i and j, which do not intersect a.

Then i and j must intersect in A.

Let b and c be the two optical lines through A in the inertia plane and let them intersect a in B and C respectively.

Let d be the second optical line through B in the inertia plane and let e be the second optical line through C in the inertia plane and let d and e intersect in D.

Then the optical lines b, c, d and e form an optical parallelogram.

Let m be the diagonal line through A and D.

Let the optical line d intersect i in E and let the optical line e intersect i in F.

Let f be the second optical line through E in the inertia plane and let g be the second optical line through F in the inertia plane and let f and g intersect in G.

Then the optical lines f, g, d and e form an optical parallelogram and the diagonal line i is of the same kind as the diagonal line a of the optical parallelogram formed by b, c, d and e.

Thus since the diagonal lines a and i do not intersect it follows by Post. XVI that the diagonal lines of the other kind to the two optical parallelograms do not intersect.

But the two optical parallelograms have the corner D in common and so they must have the diagonal line through D in common.

Thus G must lie in m.

Now suppose that the optical line d intersects j in K and that the optical line e intersects j in L.

Let k be the second optical line through K in the inertia plane and let l be the second optical line through L in the inertia plane and let k and l intersect in M.

Then the optical lines k, l, d and e form an optical parallelogram and since j is supposed not to intersect a it follows as before that M must lie in m.

But now we have the optical parallelograms formed by f, g, d and e, and by k, l, d and e having the diagonal line m in common, and so, by Post. XVI, their other diagonal lines do not intersect, which is contrary to the hypothesis that i and j intersected in A.

Thus the hypothesis that there are two general lines through A in

the inertia plane which do not intersect a leads to a contradiction and therefore is not true.

Thus there is in all cases *one single general line* through A in the inertia plane which does not intersect a.

Theorem 48

If two inertia planes P and Q have a general line a in common, and if A be any element which does not lie either in P or Q, then the inertia planes through A parallel to P and Q respectively have a general line in common.

Let R and S be the inertia planes through A parallel to P and Q respectively.

Two possibilities are open: either

(1) Q has one element at least in common with R,

or (2) Q has no element in common with R.

Consider first the case where Q has one element at least in common with R.

Here, since Q has two elements in common with P and since P and R are parallel, it follows by Theorem 46 that Q has a second element in common with R.

Further, since Q and S are parallel and R has two elements in common with Q and has the element A in common with S, it follows that R has a second element in common with S and therefore R and S have a general line, say c, in common.

Next consider the case where Q has no element in common with R.

This case has no analogue in ordinary three-dimensional geometry, but must be considered in our system which is not confined to three dimensions.

We have seen that there is at least one inertia plane containing a and another general line, say b, in R since P and R are parallel.

Let T be such an inertia plane, let B be any element in b and let U be the inertia plane through B parallel to Q.

Then, since Q and U are parallel and since T contains the general line a and also the element B of U, it follows that T contains a general line, say b', in U.

But the general lines b and b' both contain the element B and neither of them can intersect a.

Thus, since b and b' both lie in one inertia plane T, it follows by Theorem 47 that they must be identical, and so b must be common to U and R.

Now the inertia planes S and U are both parallel to Q and therefore must be either parallel to one another or else identical.

If they are not identical, the inertia plane R has the general line b in common with U and has the element A in common with S.

Thus in either case R and S have a general line in common.

If we consider case (2) of the last theorem it is clear that, if the general line a be an optical line, then since the general line b lies in the same inertia plane T and has no element in common with a, it follows by Theorem 47 that b must also be an optical line and be parallel to a.

If c be the general line common to R and S, then provided c and b are distinct, it follows in a similar manner that c is an optical line parallel to b and therefore also parallel to a.

A similar result follows in case (1) and so we always have c parallel to a provided a be an optical line.

Now we have as yet given no definition of the parallelism of any type of general lines except optical lines, but are now in a position to do so.

Definition. If a be a general line and A be any element which does not lie in it and if two inertia planes R and S through A are parallel respectively to two others P and Q containing a, then the general line which R and S have in common is said to be *parallel* to a.

Theorem 49

If a be a general line and A be any element which does not lie in it, then there is one single general line containing A and parallel to a.

Two cases have to be considered:

(1) The element A lies in an inertia plane containing a.

(2) The element A does not lie in an inertia plane containing a.

Consider first case (1) and let T be the inertia plane containing A and a.

Let P_1, P_2, P_3, P_4 be any other inertia planes containing a, and let Q_1, Q_2, Q_3, Q_4 be inertia planes through A parallel to P_1, P_2, P_3, P_4 respectively.

Then, since the inertia plane T has the general line a in common with P_1 and has the element A in common with Q_1, it follows that it has a general line, say b, in common with Q_1 and b does not intersect a.

But, by Theorem 47, there is only one general line through A in the inertia plane T which does not intersect a and so b must be this general line.

Similarly Q_2, Q_3, Q_4 must all contain the general line b in common

with T and so any pair of the inertia planes Q_1, Q_2, Q_3, Q_4 have the same general line b in common.

Thus b is independent of the particular pair of inertia planes P_1, P_2, P_3, P_4 which we may select and so there is only one general line through A parallel to a.

Suppose next that A does not lie in an inertia plane containing a and suppose that P_1, P_2, P_3, P_4 are any inertia planes which are distinct from one another and all contain a.

Let Q_1, Q_2, Q_3, Q_4 be inertia planes through A and parallel to P_1, P_2, P_3, P_4 respectively.

Let P_n be an inertia plane containing a and a general line b in Q_1.

Then b is parallel to a and lies in the same inertia plane P_n with it.

If then we take inertia planes Q_2', Q_3', Q_4' through any element of b and parallel to P_2, P_3, P_4 respectively, these will all contain b and will also be respectively parallel to Q_2, Q_3, Q_4 which contain the element A.

But the general line b and the element A lie in the inertia plane Q_1 and so, by case (1), Q_2, Q_3, Q_4 all have the same general line, say c, in common with Q_1.

Thus any pair of the inertia planes Q_1, Q_2, Q_3, Q_4 have the same general line c in common.

It follows that c is independent of the particular pair of the inertia planes P_1, P_2, P_3, P_4 which we may select and so there is only one general line through A parallel to a.

Thus the theorem holds in general.

Theorem 50

If two distinct general lines are each parallel to a third, then they are parallel to one another.

Let a and b be two distinct general lines which are each parallel to the general line c.

Let R_1 and R_2 be two inertia planes each containing c but not containing a or b.

Let P_1 and P_2 be two inertia planes parallel respectively to R_1 and R_2 and through any element of a.

Then P_1 and P_2 each contain a.

Similarly let Q_1 and Q_2 be two inertia planes parallel respectively to R_1 and R_2 and containing b.

Then Q_1 is either parallel to P_1 or identical with it, while Q_2 is either parallel to P_2 or identical with it.

In either case we must have a parallel to b.

Remarks

If a and b be any pair of parallel general lines, it is easy to see that they must be general lines of the same kind, for we know already that two parallel general lines in one inertia plane must be of the same kind, and by two applications of this result it follows that if a and b do not lie in one inertia plane they must also be of the same kind.

Theorem 51

If two parallel general lines a and b lie in one inertia plane R and if two other distinct inertia planes P and Q containing a and b respectively have an element A in common, then P and Q have a general line in common which is parallel to a and b.

Let any element in b be selected and let S be the inertia plane through this element and parallel to P.

Then the general line b must lie in S and so, since Q contains the general line b and the element A, it follows that P and Q contain a general line in common which is parallel to b and therefore also parallel to a.

Theorem 52

If a pair of non-parallel general lines a and b lie in one inertia plane P and if through an element A not lying in the inertia plane there are two other general lines c and d respectively parallel to a and b, then c and d lie in an inertia plane parallel to P.

Let R be any inertia plane distinct from P which contains a but not A, and let S be any inertia plane distinct from P which contains b but not A.

Let P' be the inertia plane through A parallel to P, while R' and S' are the inertia planes through A parallel to R and S respectively.

Then P' and R' have a general line in common which is parallel to a and since it passes through A must be identical with c; while P' and S' have a general line in common which is parallel to b and since it passes through A must be identical with d.

Thus c and d lie in the inertia plane P' which is parallel to P.

Theorem 53

If three distinct inertia planes P, Q and R and three parallel general lines a, b and c be such that a lies in P and R, b in Q and P and c in R and Q, then if Q' be an inertia plane parallel to Q through any element of P which does not lie in b the inertia planes R and Q' have a general line in common which is parallel to c.

Since the inertia plane P contains two elements in common with Q

and one element in common with the parallel inertia plane Q', it follows by Theorem 46 that P and Q' have two elements in common and therefore have a general line in common which is parallel to b. Call this general line d.

If this general line should happen to coincide with a, the result follows directly.

We shall therefore consider the case where it does not coincide with a.

Let A be any element in a.

Then, in case a be an optical line, the other optical line through A in the inertia plane P will intersect b, while, if a be an inertia or separation line, both the optical lines through A in the inertia plane P will intersect b.

Thus in all cases there is at least one optical line through A in the inertia plane P which intersects b.

Let such an optical line intersect b in B and let an optical line through B in the inertia plane Q intersect c in C.

Then BA and BC may be taken as generators of opposite sets of an inertia plane, say S, which contains A, B and C.

Now the general line a is parallel to b and therefore also parallel to d, and, since BA passes through A, is distinct from a, and lies in the inertia plane P, it follows that BA intersects d in some element, say D, which accordingly lies in the inertia plane Q'.

But since D lies in BA it lies in the inertia plane S and thus S contains two elements (B and C) in common with Q and an element D in common with the parallel inertia plane Q'.

It follows by Theorem 46 that S contains a second element in common with Q' and so S and Q' contain a general line in common which must be parallel to CB.

If we denote this general line in S and Q' by g, then any general line through C in the inertia plane S, with the exception of CB, must intersect g.

But the element A does not lie in b and so does not lie in the inertia plane Q and therefore does not lie in CB.

Thus since the general line CA is distinct from CB, and since CA must lie in S, it follows that CA must intersect g in some element, say F.

But C and A both lie in the inertia plane R which accordingly must contain the general line CA and therefore the element F.

Thus since the inertia plane R contains the general line c in common with Q and contains the element F in the parallel inertia plane Q', it follows that R must have a general line in common with Q' and this general line must be parallel to c.

Theorem 54

If P_1 and P_2 be a pair of parallel inertia planes while an inertia plane Q_1 has parallel general lines a and b in common with P_1 and P_2 respectively and if Q_2 be an inertia plane parallel to Q_1 through some element (say C) of P_2 which does not lie in b, then the inertia planes P_1 and Q_2 will have a general line in common which is parallel to a and b.

Since Q_2 is parallel to Q_1 and since P_2 has the general line b in common with Q_1 and has the element C in common with Q_2, it follows, by Theorem 46, that P_2 and Q_2 have a general line (say c) in common which is parallel to b and therefore also to a.

Let A be any element of a and let g be any inertia line in P_2 which does not coincide with either b or c, while G is the one single element common to g and the α sub-set of A. Then AG is an optical line.

Let E be any element of g which is *after* G but does not lie either in b or c.

Then AE will be an inertia line so that E and the general line a lie in an inertia plane which we shall call R.

Then, by Theorem 51, P_2 and R have a general line (say e) in common which is parallel to a, b and c.

But now, by Theorem 53, since the three distinct inertia planes P_2, Q_1 and R and the three parallel general lines e, b and a are such that e lies in P_2 and R, b in Q_1 and P_2 and a in R and Q_1, and since further Q_2 is an inertia plane parallel to Q_1 through the element C of P_2 which does not lie in b, it follows that R and Q_2 have a general line (say f) in common which is parallel to a and therefore also to e and c.

Making use of the same theorem a second time, we have the three distinct inertia planes R, P_2 and Q_2 and the three parallel general lines f, e and c such that f lies in R and Q_2, e in P_2 and R and c in Q_2 and P_2, and so, since P_1 is an inertia plane parallel to P_2 through an element of R which does not lie in e, it follows that the inertia planes Q_2 and P_1 have a general line (say d) in common which is parallel to c and therefore also parallel to a and b.

Thus the theorem is proved.

Theorem 55

(*a*) *If a and b be two parallel separation lines in the same inertia plane and if one element of b be* before *an element of a, then each element of b is* before *an element of a.*

Let A be the element of b which by hypothesis is *before* an element of a.

Let the two optical lines through A in the inertia plane be called c and d.

Let B be any other element of b.

Then by Theorem 40 B must be *before* an element of one of the optical lines c and d and *after* an element of the other.

It will be sufficient to consider the case when B is *before* an element of c and *after* an element of d, since the proof in the other case is similar.

Let e and f be the two optical lines through B in the inertia plane and let e be the one which is parallel to c. Then f intersects c in some element C.

Also c intersects a in some element D (Post. XV) and D must be *after* A; for since A is *before* an element of a, we should otherwise have one element of a *after* another, contrary to the hypothesis that a is a separation line.

Now, since B is *before* an element of c and cannot also be *after* an element of c, and since C lies in the optical line f through B, it follows that C is *after* B.

Now C cannot be *before* A for then A would be *after* B, contrary to the hypothesis that A and B lie in a separation line.

If C be either *before* D or coincident with D, then B is *before* D an element of a.

Suppose next that C is *after* D and let E be the element in which f intersects a.

Let h be the second optical line through D in the inertia plane and let g be the second optical line through E in the inertia plane and let g and h intersect in F.

Then the optical lines c, f, h and g form an optical parallelogram whose diagonal line through D and E is a.

Let j be the other diagonal line through C and F, then j is an inertia line.

Let the optical lines d and e intersect in G.

Then the optical lines c, f, d and e form an optical parallelogram whose diagonal line through A and B is b.

Thus in the two optical parallelograms, since the diagonal lines a and b do not intersect, it follows that the diagonal lines of the other kind do not intersect (Post. XVI).

But the two optical parallelograms have the corner C in common and so they must have a diagonal line in common and so G must lie in j.

Also D is *after* A and so h must be an after-parallel of d.

But, since F and G are elements of j which is an inertia line, it follows

that the one is *after* the other; and since no element of d can be *after* an element of h, it follows that F must be *after* G.

Thus since F is an element of g and G is an element of e, it follows that g is an after-parallel of e.

But since E and B lie in the optical line f, one of them must be *after* the other, and since B lies in e it cannot be *after* E which is an element of g.

Thus E is *after* B and so B is *before* an element of a.

Thus in all cases B is *before* an element of a.

(*b*) *If a and b be two parallel separation lines in the same inertia plane and if one element of b be* after *an element of a, then each element of b is* after *an element of a.*

THEOREM 56

(*a*) *If a and b be a pair of parallel separation lines in the same inertia plane and if an optical line c intersects a in A_1 and b in B_1 while a parallel optical line d intersects a in A_2 and b in B_2, then if B_1 is* before *A_1 we have also B_2* before *A_2.*

By Theorem 55, since B_1 is *before* A_1, therefore B_2 is *before* an element of a.

But since A_2 and B_2 are distinct elements in the optical line d, therefore one of them is *after* the other.

Further, B_2 could not be *after* A_2 for then since B_2 is *before* an element of a we should have A_2 *before* this element of a, contrary to the hypothesis that a is a separation line.

Thus B_2 must be *before* A_2.

(*b*) *If a and b be a pair of parallel separation lines in the same inertia plane and if an optical line c intersects a in A_1 and b in B_1 while a parallel optical line d intersects a in A_2 and b in B_2, then if B_1 is* after *A_1 we have also B_2* after *A_2.*

THEOREM 57

(*a*) *If a and b be a pair of parallel inertia lines in the same inertia plane and if an optical line c intersect a in A_1 and b in B_1, while a parallel optical line d intersects a in A_2 and b in B_2; then if B_1 is* before *A_1 we have also B_2* before *A_2.*

Since B_1 and B_2 are elements of an inertia line b, one of them must be *after* the other.

We shall first consider the case when B_2 is *after* B_1.

Let e be the second optical line through B_2 in the inertia plane and let f be the second optical line through B_1 in the inertia plane.

Then since, by hypothesis, d is parallel to c it follows that e must intersect c in some element C, while d must intersect f in some element F.

But, since B_2 is *after* B_1, it follows that e must be an after-parallel of f and d must be an after-parallel of c.

Thus, since B_1 and C lie in one optical line, it follows that C is *after* B_1 and similarly, since B_2 and C lie in one optical line, it follows that B_2 is *after* C.

Let the optical line e intersect a in D.

If then C is *before* A_1 we shall have A_1 in the α sub-set of C and by Post. XIV (b) there is *one single element* common to the inertia line a and the β sub-set of C, and since there are only two optical lines through C in the inertia plane, it follows that this element must be identical with D.

Thus D is *before* C and C is *before* B_2 and consequently D is *before* B_2 and since D and B_2 lie in one optical line it follows that D lies in the β sub-set of B_2.

If C were identical with A_1, it would also be identical with D and again D would lie in the β sub-set of B_2.

But by Post. XIV (a) there is one single element common to the inertia line a and the α sub-set of B_2 and since there are only two optical lines through B_2 in the inertia plane this element must lie in d and must therefore be identical with A_2.

Thus since A_2 lies in the α sub-set of B_2 and is not identical with B_2, therefore B_2 must be *before* A_2.

Thus in case C is either *before* A_1 or identical with A_1 we have B_2 *before* A_2.

Next suppose C is *after* A_1.

Then the optical lines e, d, c and f form an optical parallelogram whose diagonal line through B_1 and B_2 is b.

Let j be the other diagonal line through C and F.

Then since b is an inertia line, j must be a separation line.

Again let g be the second optical line through D in the inertia plane and let h be the second optical line through A_1 in the inertia plane and let g and h intersect in E.

Then the optical lines e, g, c and h form an optical parallelogram whose diagonal line through A_1 and D is a.

Thus the two optical parallelograms formed by e, d, c and f and by e, g, c and h have diagonal lines of one kind, b and a, which do not intersect and so by Post. XVI their diagonal lines of the other kind do not intersect.

But the two optical parallelograms have the corner C in common and so they have the diagonal line through C in common.

Thus E lies in j and since j is a separation line E is neither *before* nor *after* F.

But since A_1 is *after* B_1 it follows that h is an after-parallel of f and so E must be *after* an element of f.

But since E is neither *before* nor *after* F, it follows by Theorem 40 that since E is *after* an element of f it must be *before* an element of d.

Thus g is a before-parallel of d and since D and B_2 lie in the optical line e which intersects g in D and d in B_2, it follows that D is *before* B_2.

Thus D lies in the β sub-set of B_2 and in the optical line e.

But by Post. XIV (a) there is *one single element* common to the inertia line a and the α sub-set of B_2 and since there are only two optical lines through B_2 in the inertia plane it follows that this element must lie in d and is therefore identical with A_2.

Thus since A_2 is in the α sub-set of B_2 and is not identical with B_2, therefore B_2 is *before* A_2.

This proves the theorem provided B_2 is *after* B_1.

Suppose now that B_1 is *after* B_2.

Then c must be an after-parallel of d and, since A_1 and A_2 lie in c and d respectively and, since they both lie in the inertia line a, it follows that A_1 must be *after* A_2.

Suppose now, if possible, that A_2 is *before* B_2, then reversing the rôles of the inertia lines a and b it would follow from what we have already proved that, c and d being parallel, A_1 would have to be *before* B_1, contrary to hypothesis.

Thus, since B_2 must be either *after* or *before* A_2 and cannot be *after*, it follows that B_2 is *before* A_2.

(b) *If a and b be a pair of parallel inertia lines in the same inertia plane and if an optical line c intersect a in A_1 and b in B_1 while a parallel optical line d intersects a in A_2 and b in B_2; then if B_1 is* after *A_1 we have also B_2* after *A_2.*

Since a pair of parallel inertia lines always lie in an inertia plane, the words "in the same inertia plane" may be omitted in the enunciation of this theorem.

Theorem 58

If two elements A and B lie in one optical line and if two other elements C and D lie in a parallel optical line in the same inertia plane, then if A be after *B and C* after *D the general lines AD and BC intersect.*

Let a be the optical line containing A and B, and let b be the parallel optical line containing C and D.

Then the general lines AD and BC cannot be parallel optical lines, for since B is *before* A an optical line through B which intersected b would be a before-parallel of an optical line though A which intersected b and so the element in which the former optical line intersected b would be *before* the element in which the latter optical line intersected b.

Further, Theorems 56 and 57 show that AD and BC cannot be either parallel separation lines or parallel inertia lines.

Again AD and BC cannot both be optical lines for we know that two optical lines which intersect a pair of parallel optical lines are themselves parallel.

Thus we are left with the following possibilities as to the general lines AD and BC:

(1) One is an optical line and the other an inertia line.
(2) One is an optical line and the other a separation line.
(3) One is a separation line and the other an inertia line.
(4) Both are inertia lines.
(5) Both are separation lines.

In case (1) Theorem 36 shows that the general lines intersect.

In cases (2) and (3) it follows from Post. XV that the general lines intersect.

In cases (4) and (5), since we have shown that the two general lines cannot be parallel, it follows by Theorem 47 that they must intersect.

Thus in all cases the general lines AD and BC intersect.

Definitions. If four optical lines form an optical parallelogram, they will be spoken of as the *side lines* of the optical parallelogram.

A pair of side lines which do not intersect will be called *opposite.*

The element of intersection of the diagonal lines will be spoken of as the *centre* of the optical parallelogram.

Theorem 59

If any two distinct elements A and O be taken in an inertia or separation line i in a given inertia plane, then there is one single *optical parallelogram in the inertia plane having O as the centre and A as one of its corners.*

Let a and b be the two optical lines through A in the inertia plane while c and d are those through O; the optical line c being parallel to a and the optical line d parallel to b.

Let j be the second diagonal line of the optical parallelogram formed by a, b, c and d.

Then by Theorem 47 there is one single general line through O and parallel to j.

Call this general line k and let a intersect k in D while b intersects k in C.

The elements of intersection must exist since k, being parallel to j, must be an inertia or separation line according as i is a separation or inertia line; while a and b are both optical lines.

Let e be the second optical line through C in the inertia plane, while f is the second optical line through D in the inertia plane and let e and f intersect in B.

Then a, b, e and f form an optical parallelogram in the same inertia plane with that formed by a, b, c and d and their diagonal lines of one kind k and j do not intersect and so by Post. XVI their diagonal lines of the other kind do not intersect.

But the corner A is common to both optical parallelograms and so the diagonal line i which passes through that corner must be a diagonal line of both optical parallelograms.

Thus B must lie in i and so O is the centre of the optical parallelogram formed by a, b, e and f, while A is one of its corners.

Again, if there were a second optical parallelogram in the inertia plane having O as centre and A one of its corners, then such an optical parallelogram would have i as one of its diagonal lines and so the other diagonal lines of the two optical parallelograms would not intersect.

Further, since the two optical parallelograms have the element O common to these other diagonal lines, the latter must be identical.

But there are only two optical lines, a and b, through A in the inertia plane and these intersect k in D and C respectively, which must accordingly be a pair of opposite corners of the second optical parallelogram.

But then the second optical parallelogram would have e and f as its remaining side lines and so could not be distinct from the first optical parallelogram.

Thus there is no second optical parallelogram in the inertia plane having O as centre and A as one of its corners.

THEOREM 60

If two optical parallelograms have two opposite corners in common, then they have a common centre.

Two cases are possible:

(1) The common opposite corners may lie in an inertia line.

(2) The common opposite corners may lie in a separation line.

We shall consider first the case where they lie in an inertia line.

Let A and B be the two common opposite corners of the optical parallelograms: B being *after* A.

Let C and D be the other pair of opposite corners of the one optical parallelogram which we shall suppose to lie in an inertia plane P, while

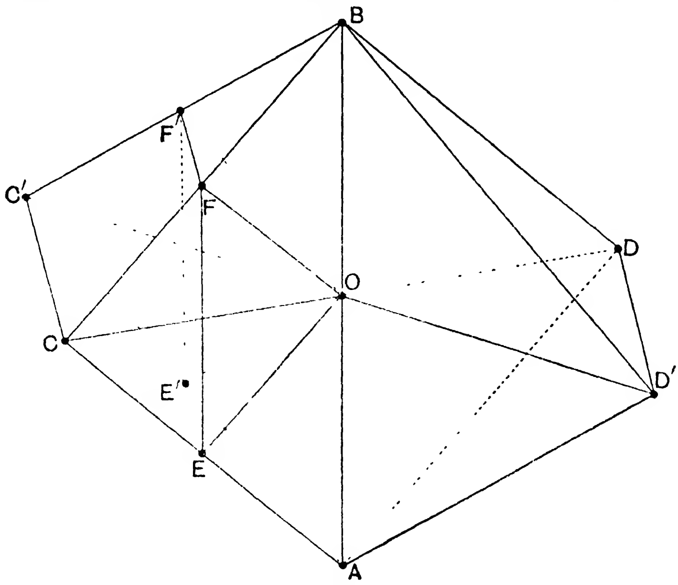


Fig. 9.

C' and D' are the other pair of opposite corners of the other optical parallelogram which we shall suppose to lie in an inertia plane P'.

Then P and P' must be distinct if the optical parallelograms are distinct.

Let O be the centre of the optical parallelogram whose corners are A, B, C, D, and let OE and OF be optical lines through O parallel to CB and AC respectively and intersecting AC and CB in E and F respectively.

Then E, C, F and O form the corners of an optical parallelogram in the inertia plane P, and this optical parallelogram and the one whose

corners are A, C, B and D have the common diagonal line CD and so their diagonal lines of the other kind do not intersect.

Thus AB and EF are parallel and EF is an inertia line.

Now let OE' and OF' be optical lines through O parallel to $C'B$ and AC' respectively and intersecting AC' and $C'B$ in E' and F' respectively.

Then AC and AC' may be taken as generators of opposite sets of an inertia plane Q_1, while OF and OF' will be generators of opposite sets of a parallel inertia plane Q_2.

Similarly BC and BC' may be taken as generators of opposite sets of an inertia plane R_1, while OE and OE' will be generators of opposite sets of a parallel inertia plane R_2.

But Q_1 and R_1 have the general line CC' in common, while Q_1 and R_2 have the general line EE' in common and so since R_1 and R_2 are parallel it follows that CC' and EE' are parallel.

Again since R_1 and Q_2 have the general line FF' in common and since Q_1 and Q_2 are parallel, it follows that FF' and CC' are parallel.

Thus FF' is parallel to EE'.

But since EF is an inertia line there exists an inertia plane containing E, F and F'. Let S be this inertia plane.

Then there exists in S a general line through E which is parallel to FF' and, since there can be only one parallel to FF' through E, this must be identical with the general line EE'.

Thus E' must lie in the inertia plane S.

But since AB and EF are parallel and lie in P while P' and S are two other distinct inertia planes containing AB and EF respectively and since P' and S have an element F' in common, it follows by Theorem 51 that the general line $E'F'$ which is common to P' and S is parallel to AB.

But now E', C', F' and O form the corners of an optical parallelogram in the inertia plane P', and this optical parallelogram and the one whose corners are A, C', B and D' have one pair of diagonal lines, namely $E'F'$ and AB, which do not intersect and so their diagonal lines of the other kind do not intersect.

But these latter diagonal lines are $C'O$ and $C'D'$ respectively and so since they have the element C' in common it follows that they are identical.

Thus the element O must lie in $C'D'$ and since it also lies in AB it follows that O is the centre of the optical parallelogram whose corners are A, C', B, D'.

Thus the optical parallelograms having A and B as opposite corners have a common centre O.

We have next to consider the case where the common opposite corners lie in a separation line.

Let A and B be the two common opposite corners of the optical parallelograms: B being neither *before* nor *after* A.

Let C and D be the other pair of opposite corners of the one optical parallelogram, which we shall suppose to lie in an inertia plane P, while C' and D' are the other pair of opposite corners of the other

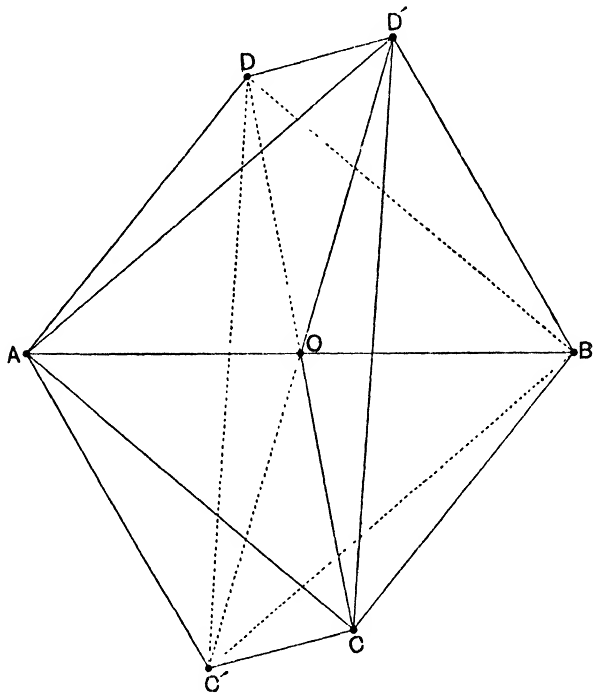


Fig. 10.

optical parallelogram, which we shall suppose to lie in an inertia plane P'.

Then P and P' must be distinct if the optical parallelograms are distinct.

We shall further suppose D to be *after* C and D' *after* C'.

Now the following pairs of intersecting optical lines may be taken as generators of opposite sets of certain inertia planes which we shall denote by the following symbols opposite each pair.

Optical lines	Inertia plane
CA and $C'A$	Q_1
BD and BD'	Q_2
CB and $C'B$	R_1
AD and AD'	R_2
AC' and AD	S_1
BD' and BC	S_2
BC' and BD	T_1
AD' and AC	T_2

Of these inertia planes we evidently have those pairs parallel which are represented by the same letters.

Thus the general line $C'D$, since it lies in S_1 and T_1, must be parallel to the general line CD', since the latter lies in S_2 and T_2.

Similarly the general line DD', since it lies in Q_2 and R_2, must be parallel to the general line $C'C$, since the latter lies in Q_1 and R_1.

But CD is an inertia line and so there is an inertia plane containing C, D and D', and if we call this inertia plane U then U contains the general lines CD' and DD' and so U must also contain the general lines through D parallel to CD' and through C parallel to DD'.

That is: the inertia plane U must contain $C'D$ and $C'C$.

Thus U must contain C' and therefore contains $C'D'$.

Thus the centres of the two optical parallelograms must lie in the inertia plane U and in the separation line AB.

The inertia plane U cannot however have more than one element in common with AB, for otherwise it would contain both A and B, and since U contains D we should have U identical with P; but U contains D' which does not lie in P and so this is impossible.

Thus the element in which CD intersects AB must be identical with the element in which $C'D'$ intersects AB, or in other words the two optical parallelograms have a common centre.

Thus the theorem is proved.

Theorem 61

If two optical parallelograms have two adjacent corners in common, then optical lines through the centres of the optical parallelograms and intersecting their common side line intersect it in the same element.

Let A and B be the two common adjacent corners of two optical parallelograms which we shall suppose to lie in separate inertia planes P and P'.

We shall suppose C and D to be the other corners of the optical parallelogram in P and shall suppose C to be opposite to B and D opposite to A.

We may further, without limitation of generality, take the diagonal line CB as the inertia diagonal line.

We shall suppose C' and D' to be the remaining corners of the optical parallelogram in P' and we shall take C' opposite to B and D' opposite to A.

Let O be the centre of the optical parallelogram in P and let the one optical line through O in the inertia plane P intersect AB in M, while the other optical line in P through O intersects AC in E.

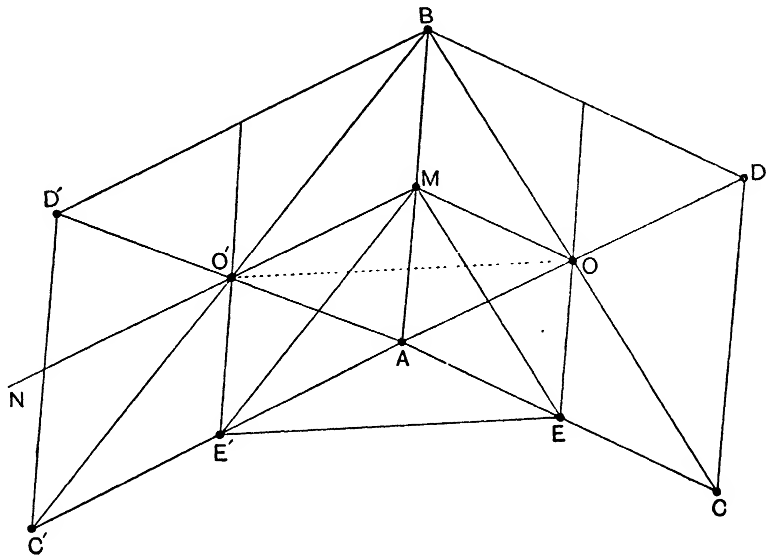


Fig. 11.

Then A, E, O and M form the corners of an optical parallelogram also in the inertia plane P.

The optical parallelograms whose corners are A, E, O, M and A, C, D, B have the diagonal line AD in common and so, by Post. XVI, their diagonal lines of the other kind do not intersect.

Thus EM and CB are parallel.

Now let MN be the optical line through M parallel to AC' and let MN intersect the diagonal line $C'B$ in O'.

Let $O'E'$ be the optical line through O' parallel to MA and intersecting AC' in E'.

Then $O'E'$ is parallel to OE and unless it be a neutral-parallel we have $O'E'$ and OE in one inertia plane.

Now, since MN is an optical line through M which neither intersects OE nor is parallel to it, it follows by Post. XII that there is *one single element* in MN which is neither *before* nor *after* any element of OE.

If O_0 be this element, we shall suppose first that O' is distinct from O_0 and thereby ensure that $O'E'$ and OE lie in one inertia plane.

Call this inertia plane Q.

Now, since MO and MO' are respectively parallel to AE and AE' and all four are optical lines, it follows that M, O and O' lie in one inertia plane, say R_1, while A, E and E' lie in a parallel inertia plane, say R_2.

But Q has the elements O and O' in common with R_1 and has the elements E and E' in common with R_2 and so the general lines OO' and EE' are parallel.

We have however further seen that OB and EM are parallel and are both inertia lines.

Thus O, O' and B lie in one inertia plane, say S_1, while E, E' and M lie in a parallel inertia plane, say S_2.

But the inertia plane P' has the elements O' and B in common with S_1 and has the elements E' and M in common with S_2.

Thus BO' and ME' are parallel.

But BO' is the same general line as BC', which is a diagonal line of the optical parallelogram whose corners are A, C', D', B, while ME' is a diagonal line of the optical parallelogram whose corners are A, E', O', M and these diagonal lines do not intersect.

It follows by Post. XVI that their other diagonal lines AD' and AO' do not intersect and so since they have the element A in common they must be identical.

Thus O' must lie in AD' and since it also lies in BC', it follows that O' is the centre of the optical parallelogram whose corners are A, C', D', B.

Thus the optical lines through the centres O and O' and intersecting AB, intersect it in the same element M.

Now this same method of proof holds for the case of any optical parallelogram in the inertia plane P' which has A and B as adjacent corners, provided that the diagonal line through B does not intersect MN in O_0, and so all such optical parallelograms have their centres in the optical line MN.

Again, if we select a second optical parallelogram in the inertia plane P having A and B as adjacent corners but not having O as centre, we may use a similar method of proof and show that all optical parallelo-

grams in the inertia plane P' having A and B as adjacent corners have, with *one possible exception*, got their centres in one optical line.

This *one possible exception* is, however, different from the *one possible exception* which we found before and so it follows that no exception exists.

Similar considerations show that all optical parallelograms in the inertia plane P, having A and B as adjacent corners, have their centres in one optical line MO.

Thus the theorem holds for optical parallelograms in the inertia planes P and P' and will therefore also hold for optical parallelograms in any other inertia planes which contain A and B.

Definition. If A and B be two distinct elements lying in an inertia line or in a separation line, then the centre of an optical parallelogram of which A and B are a pair of opposite corners will be spoken of as the *mean* of the elements A and B.

Theorem 60 shows that if two elements A and B lie in an inertia or separation line their mean is independent of the particular optical parallelogram used to define it.

Since a diagonal line of an optical parallelogram is either an inertia or a separation line, the above definition fails for the case of two distinct elements lying in an optical line.

In this case we adopt the following definition.

Definition. If A and B be two distinct elements lying in an optical line, then an optical line through the centre of an optical parallelogram of which A and B are a pair of adjacent corners and intersecting the optical line AB, intersects it in an element which will be spoken of as the *mean* of the elements A and B.

Theorem 61 shows that if two elements A and B lie in an optical line, their mean is independent of the particular optical parallelogram used to define it.

Remarks

If A, B, C and D be the corners of an optical parallelogram such that B is *after* A and C *after* B, then C will be *after* A and so AC will be the inertia diagonal line and BD will be the separation diagonal line.

If AC and BD intersect in O, then O is neither *before* nor *after* B, since O and B are elements of a separation line.

Now O cannot be *after* C, for this would entail O being *after* B, and also O cannot be *before* A, for this would entail O being *before* B.

Thus since A, O and C are distinct elements of the one inertia line we must have O *after* A and *before* C.

If now an optical line be taken through O parallel to AD and BC and intersecting AB in E, then OE must be an after-parallel of AD and a before-parallel of BC.

Thus, since A, E and B are distinct elements of the optical line AB, it follows that E is *after* A and *before* B.

We see from these results that the mean of two elements lying either in an inertia or optical line must be *after* the one and *before* the other.

Theorem 62

If A, B and B' be three distinct elements in a general line a, then the mean of A and B' must be distinct from the mean of A and B.

Let us first take the case where a is an optical line and let P be any inertia plane containing a.

Let a_1 be any optical line lying in P and parallel to a, and let optical lines through A, B and B' intersect a_1 in the elements A_1, B_1 and B_1' respectively.

Then A, A_1, B_1, B form the corners of an optical parallelogram, while A, A_1, B_1', B' form the corners of another optical parallelogram having the two adjacent corners A and A_1 in common with the first.

Let C and C' be the centres of these two optical parallelograms respectively.

Then, as we have seen, C and C' must lie in an optical line parallel to a.

An optical line through C parallel to A_1A will intersect a in some element M, which is the mean of A and B; while an optical line through C' parallel to A_1A will intersect a in some element M', which is the mean of A and B'.

Now C' cannot be identical with C, for then the general line A_1C would be identical with the general line A_1C', and so B' would have to be identical with B: contrary to hypothesis.

Thus CM and $C'M'$ must be distinct and parallel optical lines, and therefore M' must be distinct from M, as was to be proved.

Next let us consider the case where a is either an inertia line or a separation line and let P be any inertia plane containing a.

Then there is one single optical parallelogram in P having A and B as a pair of opposite corners and a centre, say C, whose position in a is independent of P by Theorem 60.

But there is also one single optical parallelogram in P having A and B' as a pair of opposite corners and a centre, say C', whose position in a is also independent of P.

Then C' could not be identical with C for, were this the case, we should have two distinct optical parallelograms in P having C as a common centre and A as a common corner, which would be contrary to what we proved in Theorem 59.

Thus, whatever type of general line a may be, the mean of A and B must be distinct from the mean of A and B'.

It follows at once from this theorem that if A and C be any two elements in any type of general line, there is not more than one element B such that C is the mean of A and B.

Theorem 63

If two or more optical parallelograms have a pair of opposite side lines in common, their centres lie in a parallel optical line in the same inertia plane.

We have already seen in the course of proving Theorem 61 that this result must hold if the two optical parallelograms have a third side in common.

In case this is not so, let A_1, B_1, C_1, D_1 be four distinct elements in an optical line a and let b be a parallel optical line in an inertia plane containing a.

Let the second optical lines through A_1, B_1, C_1, D_1 respectively in the inertia plane intersect b in A_2, B_2, C_2, D_2 respectively and let A_1, B_1, A_2, B_2 be the corners of one of the optical parallelograms under consideration and C_1, D_1, C_2, D_2 the corners of another.

Then A_1, D_1, A_2, D_2 is a third optical parallelogram.

Call these optical parallelograms (1), (2) and (3) and let their centres be O, O', O'' respectively.

Then by the first case O and O'' lie in an optical line parallel to a and b since (1) and (3) have the pair of adjacent corners A_1 and A_2 in common.

Similarly O' and O'' lie in an optical line parallel to a and b since (2) and (3) have the pair of adjacent corners D_1 and D_2 in common.

But there is only one optical line through O'' parallel to a and b and so O, O' and O'' lie in one optical line parallel to a and b.

Thus all optical parallelograms having a and b as a pair of opposite side lines must have their centres in the optical line OO'.

Theorem 64

If two optical parallelograms have a pair of opposite side lines in common and if one diagonal line of the one optical parallelogram passes through the centre of the other, then the two optical parallelograms have a common centre.

Since the centre of an optical parallelogram is the element of intersection of its diagonal lines, and since, by hypothesis, one diagonal line of the one optical parallelogram passes through the centre of the other, it follows that both centres must lie in that diagonal line.

Now we know that in any optical parallelogram the one diagonal line is an inertia line, while the other is a separation line.

Thus the centres of the two optical parallelograms must lie in an inertia line or a separation line.

But we have already seen by Theorem 63 that they lie in an optical line, and since any two distinct elements determine a general line, it follows that the centres cannot be distinct.

Thus the two optical parallelograms have a common centre.

Theorem 65

If two optical parallelograms P and Q in the same inertia plane have a common centre, then the elements in which a pair of opposite side lines of P intersect the diagonal lines of Q form the corners of an optical parallelogram with the same centre.

Let O be the common centre of the two optical parallelograms P and Q and let i and j be the two diagonal lines of Q while a and b are a pair of opposite side lines of P.

Let a intersect i in E and j in F, while b intersects i in G and j in H.

Denote the second optical line through E in the inertia plane by c, and suppose it intersects b in H'.

Denote the second optical line through G in the inertia plane by d, and suppose it intersects a in F'.

Then the optical lines a, c, b and d form an optical parallelogram one of whose diagonal lines, namely i, passes through O the centre of the optical parallelogram P of which a and b are opposite side lines, and so by Theorem 64 these two optical parallelograms have a common centre O.

Thus if j' be the second diagonal line of the optical parallelogram formed by a, c, b and d, it has the element O in common with j.

The two optical parallelograms Q and that formed by a, c, b and d have however the diagonal line i in common and thus their diagonal lines of one kind do not intersect, and so by Post. XVI their diagonal lines of the other kind do not intersect.

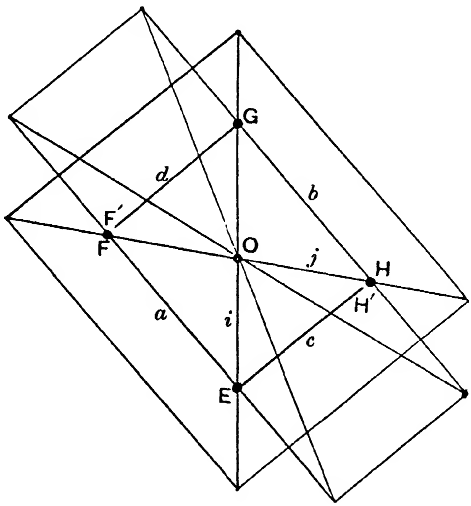


Fig. 12.

But these diagonal lines are j and j' which as we have seen have the element O in common and therefore must be identical.

Thus F' must be identical with F and H' must be identical with H and so the elements E, F, G and H must form the corners of an optical parallelogram having the same centre as the two original optical parallelograms, as was to be proved.

Remarks and Definitions

If a and b be any two distinct inertia lines and A_0 be any element in a which is not an element of intersection with b, then from Post. XIV (a) it follows that there is one single element common to the inertia line b and the α sub-set of A_0.

Call this element B_0.

Then B_0 is distinct from A_0 and cannot be an element of intersection of the two inertia lines, for if it were A_0 and B_0 would lie both in an inertia line and an optical line, which is impossible.

Further, there cannot be an element of intersection of the two inertia lines lying *after* A_0 and *before* B_0 for, by Theorem 12, any

element which is *after* A_0 and *before* B_0 must lie in the optical line A_0B_0 and so, being distinct from A_0, it could not also lie in the inertia line a.

Thus any element of intersection of the two inertia lines, if such an element exists, must lie either *before* A_0 or *after* B_0.

Again from Post. XIV (a) it follows that there is one single element, say A_1, common to the inertia line a and the α sub-set of B_0, and again A_1 cannot be an element of intersection of the inertia lines.

Further, any such element, if it exists, must lie either *before* A_0 or *after* A_1.

Proceeding again in the same way there is one single element, say B_1, common to the inertia line b and the α sub-set of A_1 and one single element A_2 common to the inertia line a and the α sub-set of B_1, and so on.

Thus we get an infinite series of elements $A_0, A_1, A_2, A_3, \ldots$ in the inertia line a and another infinite series of elements B_0, $B_1, B_2, B_3, \ldots$ in the inertia line b.

An element of intersection of the two inertia lines if such an element exists must lie either *before* A_0 or *after* A_n, where n is any finite integer whatever.

This process will be spoken of as *taking steps along the inertia line a with respect to the inertia line b.*

The passing from A_0 to A_1 is the first step, the passing from A_1 to A_2 the second, and so on.

If X be an element which is *after* A_0 in the inertia line a and *before* A_n but not *before* A_{n-1}, then the element X will be said to be *surpassed* from A_0 in n steps taken with respect to b.

If C be an element of intersection of the two inertia lines and if C be *after* A_0, it is evident from what we have said that *C cannot be surpassed from* A_0 *in any finite number of steps.*

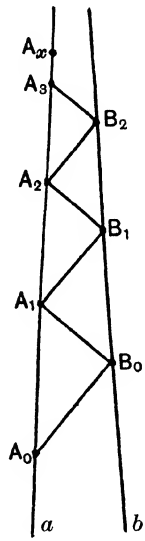


Fig. 13.

These remarks and definitions prepare the way for Post. XVII.

POSTULATE XVII. **If A_0 and A_x be two elements of an inertia line a such that A_x is after A_0, and if b be a second inertia line which does not intersect a either in A_0, A_x or any element both after A_0 and before A_x, then A_x may be surpassed in a finite number of steps taken from A_0 along a with respect to b.**

This postulate will be found to take the place of the well-known

axiom of Archimedes, to which it will be seen to bear a certain resemblance.

It, however, unlike the axiom of Archimedes, contains no reference to congruence.

It follows directly from Post. XVII that if the two inertia lines a and b *do not intersect at all* then A_x may *always* be surpassed in a finite number of steps.

There is also what is equivalent to a (b) form of this postulate which, however, is not independent.

It may be stated and proved as follows:

If A_0 and A_x be two elements of an inertia line a such that A_x is before *A_0, and if b be a second inertia line which does not intersect a either in A_0, A_x, or any element both* before *A_0 and* after *A_x, then A_0 may be reached in a finite number of steps taken along a from an element* before *A_x in a and with respect to b.*

By Post. XVII since A_0 is after A_x it follows that A_0 may be surpassed in a finite number of steps, say n, taken from A_x along a with respect to b.

Let the elements marking these steps in a be denoted by A_{x+1}, A_{x+2}, A_{x+3}, ... A_{x+n} and let the elements in b lying in the β sub-sets of these be denoted respectively by B_x, B_{x+1}, B_{x+2}, ... B_{x+n-1}.

Then A_0 may either coincide with A_{x+n-1} or be *after* it.

If A_0 coincides with A_{x+n-1}, then it is reached in $n-1$ steps taken along a from A_x.

Now there is *one single element*, say B_{x-1}, common to the inertia line b and the β sub-set of A_x and also *one single element*, say A_{x-1}, common to the inertia line a and the β sub-set of B_{x-1}.

Then A_{x-1} is *before* A_x and A_0 is reached in n steps taken along a from A_{x-1} with respect to b.

This proves the result if A_0 coincides with A_{x+n-1}.

Suppose next that A_0 does not coincide with A_{x+n-1}.

Then A_0 is *after* A_{x+n-1} and *before* A_{x+n}.

Let B_{-1} be the *one single element* common to the inertia line b and the β sub-set of A_0 and let A_{-1} be the *one single element* common to the inertia line a and the β sub-set of B_{-1}.

Let B_{-2} be the *one single element* common to the inertia line b and the β sub-set of A_{-1} and let A_{-2} be the *one single element* common to the inertia line a and the β sub-set of B_{-2}, and so on, till we get to an element A_{-n}.

Now B_{-1} cannot coincide with B_{x+n-1} for then A_0 and A_{x+n} would be two distinct elements of the inertia line a both lying in the α sub-set of B_{-1}, contrary to Post. XIV (a).

Further, A_0 is *before* A_{x+n} and B_{-1} is *before* A_0 and so B_{-1} is *before* A_{x+n}.

It follows that B_{-1} cannot be *after* B_{x+n-1}, since otherwise, by Theorem 12, B_{-1} would require to lie in the optical line $A_{x+n}B_{x+n-1}$. But B_{-1} is distinct from B_{x+n-1} and both lie in the inertia line b and therefore cannot both lie in one optical line.

It follows that B_{-1} must be *before* B_{x+n-1}.

Similarly B_{x+n-2} must be *before* B_{-1}.

Reversing the rôles of a and b we get in an analogous way:

A_{-1} is *before* A_{x+n-1} and A_{x+n-2} is *before* A_{-1}.

Repeating this reasoning we get:

A_{-2} is *before* A_{x+n-2} and A_{x+n-3} is *before* A_{-2},

....

....

and so we see that A_{-n} is *before* A_x and A_x is *before* A_{-n+1}.

Thus A_{-n} is an element in a which is *before* A_x, and A_0 may be reached in a finite number n of steps taken from A_{-n} with respect to b along a.

Thus the result holds in general.

Theorem 66

(a) *If A_0 and A_x be two elements in an inertia line a which lies in the same inertia plane with another inertia line b which does not intersect a in A_0, A_x, or any element* after *the one and* before *the other, and if an optical line through A_0 intersects b in B_0 so that B_0 is* after *A_0, then a parallel optical line through A_x will intersect b in an element which is* after *A_x.*

We shall first suppose that A_x is *after* A_0.

Let the optical line through A_x parallel to A_0B_0 intersect b in B_x.

Then by Post. XVII A_x may be surpassed in a finite number of steps say n, taken from A_0 along a with respect to b.

Let the elements (including A_0) marking these steps in a be A_0, A_1, A_2, ... A_n and let the elements in b lying in the α sub-sets of these be B_0, B_1, B_2, ... B_n respectively.

Then A_x may either coincide with A_{n-1} or be *after* it.

Now the optical line B_0A_1 intersects the two optical lines A_0B_0 and

A_1B_1 and so these latter two optical lines belong to one set and are therefore parallel.

Similarly A_1B_1 intersects the two optical lines B_0A_1 and B_1A_2 and so these two are also parallel but belong to the other set.

Proceeding thus we see that the optical lines A_0B_0, A_1B_1, A_2B_2, ... A_nB_n belong to one set and are all parallel, while B_0A_1, B_1A_2, B_2A_3, ... $B_{n-1}A_n$ belong to the other set and are all parallel.

But	A_1 lies in the α sub-set of		B_0,	
	B_1	,,	,,	A_1,
	A_2	,,	,,	B_1,
	B_{n-1}	,,	,,	A_{n-1},
	A_n	,,	,,	B_{n-1},
	B_n	,,	,,	A_n.

Thus if A_x coincides with A_{n-1}, then B_x must coincide with B_{n-1} and therefore B_x must lie in the α sub-set of A_x, and since B_x and A_x are distinct it follows that B_x is *after* A_x and the optical lines A_0B_0 and A_xB_x are parallel.

This proves the theorem in this case.

If A_x does not coincide with A_{n-1}, then it must be *after* A_{n-1} and *before* A_n.

Also since A_xB_x is parallel to A_0B_0 it must be parallel to $A_{n-1}B_{n-1}$ and to A_nB_n.

But since A_x is *after* A_{n-1} and *before* A_n it follows that A_xB_x is an after-parallel of $A_{n-1}B_{n-1}$ and a before-parallel of A_nB_n.

Further, A_xB_x must intersect the optical line $B_{n-1}A_n$ in some element, say C, since $B_{n-1}A_n$ is an optical line of the opposite set to A_xB_x and so C must be *after* B_{n-1} and *before* A_n.

Thus B_{n-1} must lie in the β sub-set of C, while A_n lies in the α sub-set of C.

But by Post. XIV (a) there is one single element common to the inertia line b and the α sub-set of C and this must lie in the other optical line through C in the inertia plane; that is to say in the optical line A_xB_x and must therefore be identical with B_x.

Similarly by Post. XIV (b) there is *one single element* common to the inertia line a and the β sub-set of C and this must be identical with A_x.

Thus C is *after* A_x and *before* B_x and therefore B_x is *after* A_x.

Thus the theorem is proved for all cases in which A_x is *after* A_0.

A similar method shows that the theorem is true when A_x is *before* A_0 except that the corresponding (b) form takes the place of Post. XVII.

Thus the theorem holds in general.

(*b*) *If* A_0 *and* A_x *be two elements in an inertia line a which lies in the same inertia plane with another inertia line b which does not intersect a in* A_0, A_x, *or any element* before *the one and* after *the other, and if an optical line through* A_0 *intersects b in* B_0 *so that* B_0 *is* before A_0, *then a parallel optical line through* A_x *will intersect b in an element which is* before A_x.

THEOREM 67

(*a*) *If* A_0 *and* A_x *be two elements in a separation line a which lies in the same inertia plane with another separation line b which does not intersect a in* A_0, A_x *or any element lying between a pair of parallel optical lines through* A_0 *and* A_x *in the inertia plane, and if an optical line through* A_0 *intersects b in* B_0 *so that* B_0 *is* after A_0, *then a parallel optical line through* A_x *will intersect b in an element which is* after A_x.

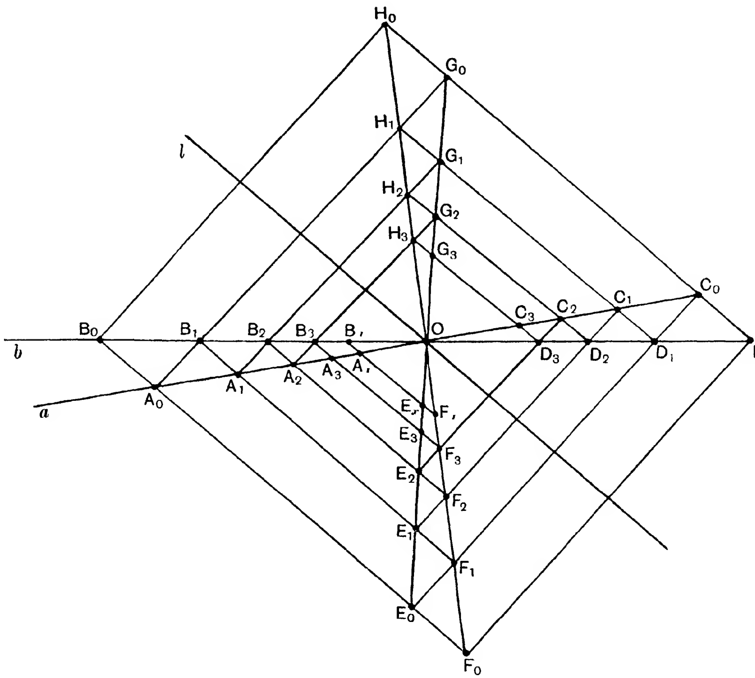


Fig. 14.

In case the separation lines *a* and *b* do not intersect at all, then since they lie in one inertia plane they are parallel and the result follows directly from Theorem 56 (*b*).

We shall therefore consider the case in which an element of intersection of *a* and *b* does exist and we shall denote this element by *O*.

We shall suppose first that A_x is between a pair of parallel optical lines through A_0 and O in the inertia plane.

Now let l be the optical line through O parallel to A_0B_0.

It will be sufficient to consider the case where l is an after-parallel of A_0B_0, since the case of a before-parallel is quite analogous.

If A_xB_x be the optical line through A_x parallel to l and meeting b in the element B_x, then A_xB_x will be an after-parallel of A_0B_0 and a before-parallel of l.

Now, by Theorem 59, there exists a definite optical parallelogram in the inertia plane having O as centre and B_0 as one of its corners, so that b is one of its diagonal lines.

Let D_0 be the corner opposite B_0 and let the optical line A_0B_0 intersect the other diagonal line in F_0 while the second optical line through B_0 in the inertia plane intersects the same diagonal line in H_0.

Then B_0, F_0, D_0 and H_0 are the corners of the optical parallelogram.

Let the separation line a intersect the optical line D_0H_0 in C_0 and let the optical line through C_0 parallel to D_0F_0 intersect B_0F_0 in E_0, while the optical line through A_0 parallel to D_0F_0 intersects D_0H_0 in G_0.

Then A_0, E_0, C_0 and G_0 are the corners of an optical parallelogram having a pair of opposite side lines in common with the optical parallelogram whose corners are B_0, F_0, D_0 and H_0 and having its diagonal line a passing through O the centre of this optical parallelogram, and so, by Theorem 64, the two optical parallelograms have a common centre O.

Denote the optical parallelogram whose corners are B_0, F_0, D_0 and H_0 by P_0 and the one whose corners are A_0, E_0, C_0 and G_0 by Q_0.

Suppose now that the optical line A_0G_0 intersects the diagonal line B_0D_0 in B_1 and the diagonal line F_0H_0 in H_1 and that the optical line E_0C_0 intersects the diagonal line F_0H_0 in F_1 and the diagonal line B_0D_0 in D_1.

Then by Theorem 65, B_1, F_1, D_1 and H_1 form the corners of an optical parallelogram having also the centre O. Call it P_1.

Suppose now that the optical line B_1F_1 intersects the diagonal line A_0C_0 in A_1 and the diagonal line E_0G_0 in E_1 and that the optical line D_1H_1 intersects the diagonal line A_0C_0 in C_1 and the diagonal line E_0G_0 in G_1.

Then, by Theorem 65, A_1, E_1, C_1, G_1 form the corners of an optical parallelogram Q_1 which bears the same relation to the optical parallelogram P_1 whose corners are B_1, F_1, D_1 and H_1 as the optical parallelogram Q_0 to the optical parallelogram P_0.

This construction may be repeated indefinitely, and we obtain a series of parallel optical lines B_0F_0, B_1F_1, B_2F_2, B_3F_3, etc., intersecting the separation line b in the elements B_0, B_1, B_2, B_3, etc., and the other diagonal line of the optical parallelogram P_0 in the elements F_0, F_1, F_2, F_3, etc.

Further, these same optical lines intersect the separation line a in the elements A_0, A_1, A_2, A_3, etc., and the other diagonal line of the optical parallelogram Q_0 in the elements E_0, E_1, E_2, E_3, etc.

Again we have another set of parallel optical lines A_0B_1, A_1B_2, A_2B_3, A_3B_4, etc., and a further set E_0F_1, E_1F_2, E_2F_3, E_3F_4, etc.

Now by hypothesis l is an after-parallel of A_0B_0 and, since OF_0 is an inertia line, it follows that F_0 is *before* O.

Similarly E_0 is *before* O.

But since b is a separation line and B_0 is *after* A_0 we must also have B_1 *after* A_0.

It follows that B_1F_1 is an after-parallel of B_0F_0 and, since E_0F_1 is an optical line, we must have F_1 *after* E_0, so that F_1 lies in the α sub-set of E_0.

Also since F_0F_1 is an inertia line we must have F_1 *after* F_0 so that F_0 is not an element of the optical line E_0F_1 but is *before* an element of it.

Thus, since F_0E_0 is an optical line, we must have E_0 *after* F_0 and so E_0 must lie in the α sub-set of F_0.

Also, from what we showed on p. 103, the element O of intersection of the two inertia lines F_0H_0 and E_0G_0 cannot lie *before* F_1 and, since we already know that F_0 is *before* O, it follows that F_1 is also *before* O.

Thus the optical line l must be an after-parallel of B_1F_1: that is to say l is an after-parallel of A_1B_1 and so E_1 is also *before* O.

But we saw that B_1 must be *after* A_0 and so since a is a separation line we must have B_1 *after* A_1.

By repetition of this reasoning we can show that:

B_2F_2	is an after-parallel of	B_1F_1
B_3F_3	,, ,,	B_2F_2
....	....	....
....	....	....

while the optical line l is an after-parallel of all these.

Also we can show that

E_1	lies in the α sub-set of	F_1	while F_2	lies in the α sub-set of	E_1
E_2	,, ,,	F_2	,, F_3	,, ,,	E_2
....	....	....	....	....	....
....	....	....	....	....	....

Thus F_1, F_2, F_3, ... mark steps taken along the inertia line F_0O with respect to the inertia line E_0O.

Now let the optical line A_xB_x intersect F_0O in F_x and E_0O in E_x.

Then, by hypothesis, A_xB_x is a before-parallel of l and it follows that both F_x and E_x are *before* O.

Thus, by Post. XVII, F_x may be surpassed in a finite number n of steps taken from F_0 along F_0O with respect to E_0O.

Now we have

$$\begin{array}{l} B_1 \text{ after } A_1, \\ B_2 \quad ,, \quad A_2, \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \end{array}$$

If then F_x should happen to coincide with F_{n-1} we should have A_x coinciding with A_{n-1} and B_x coinciding with B_{n-1} and accordingly we should have B_x *after* A_x.

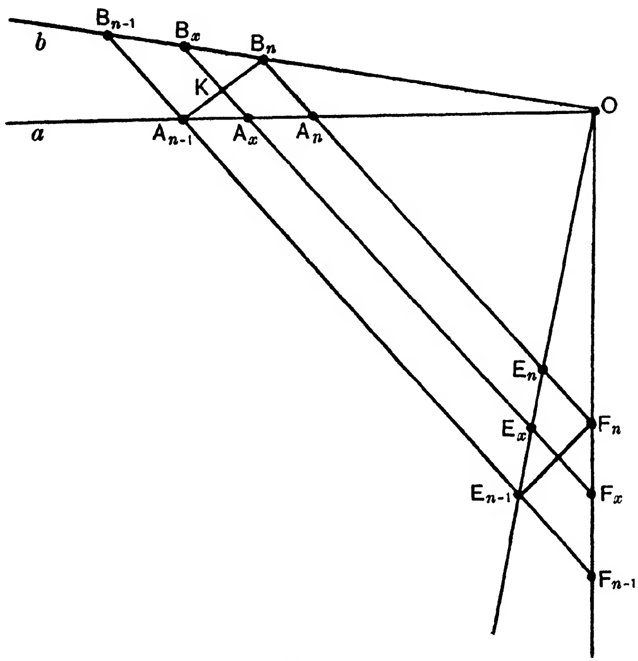


Fig. 15.

Suppose next that F_x does not coincide with F_{n-1}, but is *after* F_{n-1} and *before* F_n.

Then B_xF_x will be an after-parallel of $B_{n-1}F_{n-1}$ and a before-parallel of B_nF_n.

Let $B_x F_x$ intersect $A_{n-1} B_n$ in the element K.

Then, since $A_{n-1} K$ is an optical line, we must have K *after* A_{n-1} and also K *before* B_n.

Since A_{n-1} and A_x lie in the separation line a, we must have K *after* A_x: while, since B_n and B_x lie in the separation line b, we must have B_x *after* K.

It follows by Post. III that B_x must be *after* A_x as was to be proved.

Now we started out by considering the case where A_x is between a pair of parallel optical lines through A_0 and O in the inertia plane; if instead we had taken the case where A_0 is between a pair of parallel optical lines through A_x and O in the inertia plane, then the supposition that A_x was *after* B_x would, in a similar manner, lead to the conclusion that A_0 was *after* B_0, contrary to the hypothesis that B_0 is *after* A_0.

Also, since A_x and B_x could not coincide without the separation lines being identical, it follows that we must also in this case have B_x *after* A_x.

Thus the theorem holds in general.

(*b*) *If A_0 and A_x be two elements in a separation line a which lies in the same inertia plane with another separation line b which does not intersect a in A_0, A_x or any element lying between a pair of parallel optical lines through A_0 and A_x in the inertia plane, and if an optical line through A_0 intersects b in B_0 so that B_0 is* before *A_0, then a parallel optical line through A_x will intersect b in an element which is* before *A_x.*

THEOREM 68

If two elements A and B lie in one optical line and if two other elements C and D lie in a parallel optical line in the same inertia plane, then if A be after *B and C* after *D the element of intersection of the general lines AD and BC (which was proved in Theorem* 58 *to exist) lies between the two given optical lines.*

Let a be the optical line containing A and B, and let b be the parallel optical line containing C and D.

Since one of the optical lines must be an after-parallel of the other and since it is immaterial which of them, we shall suppose that a is an after-parallel of b.

Now the general lines AD and BC cannot *both* be optical lines since two optical lines which intersect a pair of parallel optical lines are themselves parallel and have no element of intersection.

One of them however may be an optical line.

Suppose first that BC is an optical line and that E is the element of intersection of AD and BC.

Then, since a is an after-parallel of b and CB is an optical line, therefore B is *after* C.

But C is *after* D and therefore B is *after* D, and since A is *after* B it follows that A is *after* D.

Thus, since AD cannot be an optical line and has one element which is *after* another, it must be an inertia line.

Now, since C is *after* D and lies in an optical line containing D, it follows that D is in the β sub-set of C; and since E lies in the second optical line through C in the inertia plane, it follows by Post. XIV (a) that E must be in the α sub-set of C.

Thus, since E cannot be identical with C, it follows that E is *after* C.

Similarly, since A is *after* B and A and B lie in an optical line, it follows that A is in the α sub-set of B; and since E lies in the second optical line through B in the inertia plane, it follows by Post. XIV (b) that E must be in the β sub-set of B.

Thus since E cannot be identical with B, it follows that E is *before* B.

This proves that E lies between a and b.

Suppose secondly that AD is an optical line and again let E be the element of intersection of AD and BC.

Let the optical line through C parallel to DA intersect a in F.

Then C being *after* D it follows that F must be *after* A and since A is *after* B therefore F must be *after* B.

Now F must be *after* C and therefore C lies in the β sub-set of F, as does also B.

But if C were either *before* or *after* B then, by Theorem 13 (b), C would have to lie in a which is impossible.

Thus C is neither *before* nor *after* B, so that CB must be a separation line.

Now D cannot be *after* E, for since C is *after* D we should then have C *after* E which is impossible since C and E lie in a separation line.

But since D and E are distinct elements of an optical line, the one must be *after* the other and thus E must be *after* D.

Again E cannot be *after* A, for since A is *after* B we should then have E *after* B which is impossible since E and B are elements of a separation line.

But E must be either *before* or *after* A since E and A are distinct elements of an optical line, and since E cannot be *after*, it must be *before* A.

Thus again in this case E lies between a and b.

Next take the case where one of the two general lines AD and BC is an inertia line and the other a separation line.

If BC be a separation line and E be the element of intersection with AD, then E is neither *before* nor *after* C and also neither *before* nor *after* B.

But E cannot be *before* D, for since D is *before* C we should then have C *after* E, which is impossible.

Thus since D and E are distinct elements of an inertia line, we must have E *after* D.

Again E cannot be *after* A, for since A is *after* B we should then have E *after* B, which is impossible.

Thus since A and E are distinct elements of an inertia line, we must have E *before* A.

Thus again in this case E lies between a and b.

If BC is an inertia line we must have B *after* C, since a is an after-parallel of b.

Since then C is *after* D we must have B *after* D, and since A is *after* B we must have A *after* D.

But AD could not be an optical line, for, since B is *after* D and *before* A, it would then follow by Theorem 12 that B must itself be an element of AD; which is impossible. Thus AD must be an inertia line.

Accordingly we shall next take the case where both the general lines AD and BC are inertia lines and E is their element of intersection.

By Theorem 66, if A were *before* E then C being *after* D would imply that B was *after* A, contrary to hypothesis; while if D were *after* E then A being *after* B would imply that D was *after* C, contrary again to hypothesis.

Thus since E cannot be identical with either A or D, it follows that E must be *after* D and *before* A and so E lies between a and b.

Finally we have the case where AD and BC are both separation lines and E their element of intersection.

Let c be an optical line through E parallel to a and b.

First suppose, if possible, that c is an after-parallel of a; then c would also be an after-parallel of b since a is an after-parallel of b.

Thus AD and BC would intersect in an element which was not between a and b and did not lie either in a or b, and so by Theorem 67, A being *after* B would imply that D was *after* C, contrary to hypothesis.

The same would hold if we supposed c to be a before-parallel of b.

Thus c cannot be an after-parallel of a and cannot be identical with a and therefore must be a before-parallel of a.

Also c cannot be a before-parallel of b and cannot be identical with b, and thus c must be an after-parallel of b.

Thus the element E must be *after* an element of b and *before* an element of a and so E lies between a and b.

This exhausts all the possibilities and so we see that the theorem holds in general.

Theorem 69

If two elements A and B lie in one optical line and if two other elements C and D lie in a parallel optical line in the same inertia plane, then if A be after *B and if the general lines AD and BC intersect in an element E lying between the parallel optical lines, we must also have C* after *D.*

Let a be the optical line containing A and B, and let b be the parallel optical line containing C and D.

Then one of the optical lines a and b is an after-parallel of the other, but as the demonstration is quite analogous in the two cases we shall only consider that in which a is an after-parallel of b.

We must therefore have E *after* an element of b and *before* an element of a.

Now AD and BC cannot both be optical lines since two optical lines which both intersect a pair of parallel optical lines are themselves parallel and so the element E could not exist.

We may however have one of them an optical line and shall first consider the case in which AD is such.

In this case E is *before* A and therefore E lies in the β sub-set of A, as does also B.

But E cannot be either *after* or *before* B, for otherwise, by Theorem 13 (b), E would require to lie in the optical line a and so E could not lie between a and b.

It follows that BE must be a separation line.

Thus C can be neither *before* nor *after* E.

But D is *before* E and so if C were *before* D we should have C *before* E, which is impossible.

Further, C cannot coincide with D and therefore C must be *after* D.

We shall next consider the case where BC is an optical line.

Then we have B *after* E, and since A is *after* B it follows that A is *after* E and so AE is an inertia line.

Again E is *after* C and so E lies in the α sub-set of C and therefore by Post. XIV (b) D must lie in the β sub-set of C.

Thus since C and D cannot be identical, we must have C *after* D.

We shall next consider the case where one of the general lines BC and AD is an inertia line and the other a separation line.

Now if BC were an inertia line we should have B *after* E and so, since A is *after* B, we should have also A *after* E.

Thus in this case both general lines would be inertia lines and so we must suppose instead that BC is a separation line and AD an inertia line.

Then since E cannot be *before* any element of b, and since it must be either *before* or *after* D it follows that E must be *after* D.

But D cannot be *after* C, for then we should have E *after* C, which is impossible since C and E lie in a separation line.

Thus since C and D cannot be identical, we must have C *after* D.

We have next to consider the cases where the general lines BC and AD are both separation lines and where they are both inertia lines.

The constructions and demonstrations are analogous in both cases up to a certain point.

By Theorem 59 there is an optical parallelogram in the inertia plane having E as centre and B as one of its corners.

Let C' be the corner opposite to B and let the optical line through C' in the inertia plane and of the opposite set to AB intersect AB in the element G.

Then GE is the other diagonal line of the optical parallelogram.

Let the second optical line through B in the inertia plane intersect GE in F.

Then B, F, C' and G are the corners of the optical parallelogram.

Let AE intersect the optical line FC' in D'; let an optical line through A parallel to BF intersect FC' in H, and let an optical line through D parallel to $C'G$ intersect BG in I.

Then A, H, D' and I are the corners of an optical parallelogram having a pair of opposite side lines in common with the optical parallelogram whose corners are B, F, C' and G and having one of its diagonal lines AD' passing through E the centre of this optical parallelogram.

It follows from Theorem 64 that these two optical parallelograms have a common centre.

Let AH intersect BC' in A_1 and FG in F_1 and let ID' intersect BC' in C_1 and FG in G_1.

Then by Theorem 65 the elements A_1, F_1, C_1 and G_1 form the corners of another optical parallelogram with the same centre.

Suppose now first that AE and BE are both separation lines, then EG and EI are both inertia lines, and by hypothesis E is *before* an element of BG and so E must be *before* G and also *before* I.

Also, since B and A_1 lie in a separation line and since A is *after* B, it follows that A must also be *after* A_1.

Thus A_1G_1 must be a before-parallel of BG and so G_1 must be *before* G.

Thus G_1D' must be a before-parallel of GC', and since C' and D' lie in an optical line we must have C' *after* D'.

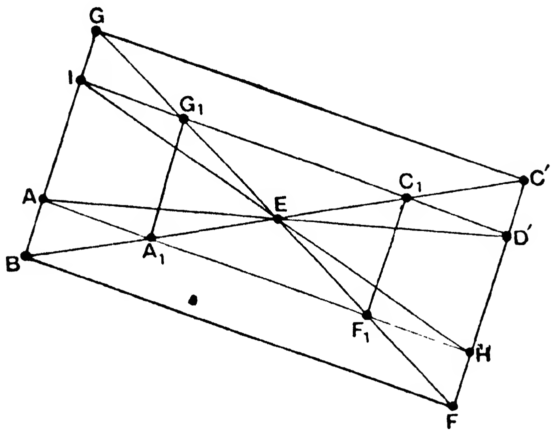


Fig. 16.

Now E being the centre of the optical parallelogram whose corners are B, G, C' and F and being *before* an element of BG must be *after* an element of FC'.

Thus E is between the parallel optical lines BG and FC'.

Now the optical line b containing C and D may either coincide with FC', in which case C is *after* D, or else b may be a before-parallel of FC' or an after-parallel of FC'.

In any case, however, if e be an optical line through E parallel to a and b, then FC' and b are each before-parallels of e, so that in no case can E lie between FC' and b.

Thus by Theorem 67 since C' is *after* D' we must have C *after* D.

Suppose next that AE and BE are both inertia lines, then EG and EI are both separation lines, and by hypothesis E is *before* an element of BG, so E is *before* A and also *before* B.

Also, since B and A_1 lie in an inertia line and since B is in the β sub-set

of A and distinct from it, therefore A_1 must be in the α sub-set of A, and since B and A are distinct, A and A_1 must also be distinct and therefore A_1 is *after* A.

Thus A_1G_1 must be an after-parallel of AI, and since G_1 and I lie in an optical line we must have G_1 *after* I.

But since G_1 and G lie in a separation line, the one is neither *before* nor *after* the other and so G must also be *after* I.

Thus GC' must be an after-parallel of ID', and since C' and D' lie in an optical line we must have C' *after* D'.

From this point the demonstration is similar to that of the case where AE and BE are both separation lines, except that the reference is to Theorem 66 instead of Theorem 67.

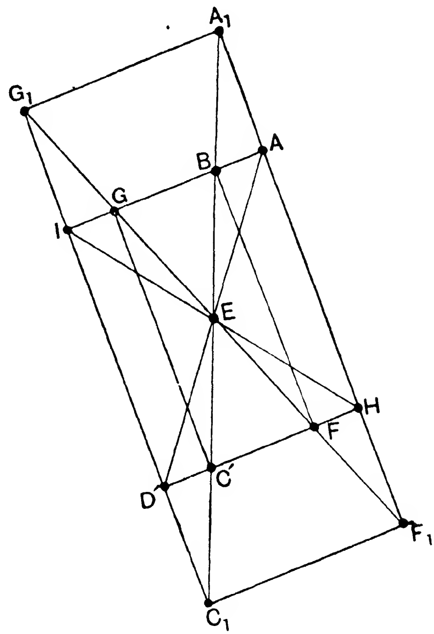


Fig. 17.

This exhausts all the possibilities, and so the theorem holds in general.

Theorem 70

If A, B and C be three elements in a separation line and if B be between a pair of parallel optical lines through A and C in an inertia plane containing the separation line, then B is also between a pair of parallel optical lines through A and C in any other inertia plane containing the separation line.

Let a be an optical line through A, and c a parallel optical line through C; both lying in the given inertia plane, say P, and such that B lies between a and c.

We may suppose that B is *before* an element of a and *after* an element of c without any essential loss of generality.

Let an optical line through B in the inertia plane, and of the opposite set to a and c, intersect a in D and c in E.

Then D must be *after* B, and E must be *before* B.

Further, since A, B and C lie in a separation line, we must have D *after* A and E *before* C.

Now let Q be any other inertia plane containing the separation line, and let a', b' and c' be three parallel optical lines through A, B and C respectively in the inertia plane Q.

Now the element D is *after* B, an element of the optical line b', while the optical line a passes through D but does not intersect b', since then it would have to be identical with the optical line DB which belongs to the opposite set.

Further, the optical line a cannot be parallel to b', for since a passes through A it would in that case have to be identical with a' and the inertia planes P and Q could not be distinct.

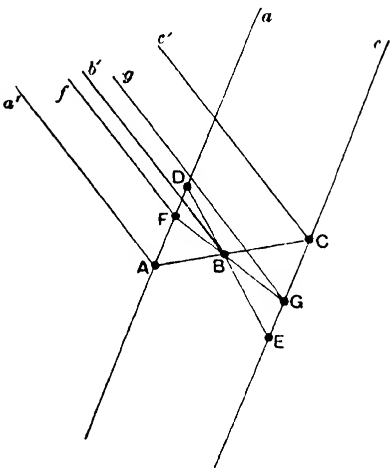


Fig. 18.

Thus each element of a is not *after* an element of b', and so by Post. XII (b) there is one single element of a, say F, which is neither *after* nor *before* any element of b'.

Thus by Theorem 22 there is one single optical line containing F and such that no element of it is either *before* or *after* any element of b'.

If f be this optical line, then f is a neutral-parallel of b'.

But since a' and b' lie in the inertia plane Q and are parallel, the one must be an after-parallel of the other and so a' cannot be identical with f.

Thus F must be either *after* or *before* A and cannot be identical with it.

Now the general line FB lies in the inertia plane P and is clearly a separation line since F is neither *before* nor *after* B.

Let FB intersect the optical line c in G.

Then, by Theorem 45, G is neither *before* nor *after* any element of b', and so if an optical line g be taken through G parallel to b' it will be a neutral-parallel.

Now, by Theorem 69, since B lies between the parallel optical lines a and c passing through A and C respectively and lying in the inertia plane P, it follows that if F be *after* A then C is *after* G; while if A be *after* F then G is *after* C.

If however F be *after* A, then a' must be a before-parallel of f, and therefore, by Theorem 26 (a), a' must be a before-parallel of b'.

Then we shall have also c' an after-parallel of g, and therefore, by Theorem 26 (b), c' must be an after-parallel of b'.

Thus B will be *after* an element of a' and *before* an element of c': that is, B will be between the parallel optical lines a' and c' passing through A and C respectively in the inertia plane Q.

Similarly if F be *before* A, then a' must be an after-parallel of f, and therefore, by Theorem 26 (b), a' must be an after-parallel of b'.

We shall in that case have also c' a before-parallel of g, and therefore, by Theorem 26 (a), c' must be a before-parallel of b'.

Thus again we shall have B between the parallel optical lines a' and c' passing through A and C respectively in the inertia plane Q.

Thus the theorem is proved.

Remarks

If A, B and C be three elements in an optical or inertia line l, and if B be between a pair of parallel optical lines through A and C in an inertia plane containing l, then it is easy to see that B is also between a pair of parallel optical lines through A and C in any other inertia plane containing l.

This follows directly from the consideration that, in this case, of any two of the three elements A, B, C, one is *after* the other.

We accordingly introduce the following definition.

Definition. If three distinct elements lie in a general line and if one of them lies between a pair of parallel optical lines through the other two in an inertia plane containing the general line, then the element which is between the parallel optical lines will be said to be *linearly between* the other two elements.

The above definition is so framed as to apply to all three types of general line and for this reason is more complicated than it need be if we were dealing only with optical or inertia lines.

For the case of elements lying in either of these types of general line, one element is linearly between two other elements if it be *after* the one and *before* the other.

In the case of elements lying in a separation line, however, no one is either *before* or *after* another and so we have to fall back on our definition involving parallel optical lines.

The distinction between the three cases is interesting.

Thus if the three elements A, B and C lie in a general line a, and if B

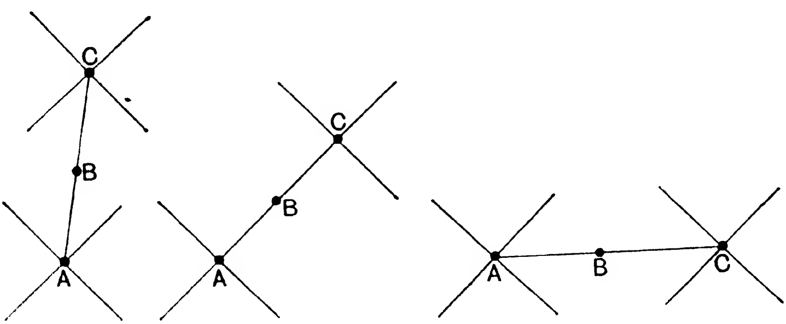
be linearly between A and C, then, in case a be an inertia line, we must have either B *after* A and C *after* B or else B *after* C and A *after* B, and similarly when a is an optical line.

If a be an inertia line and B be *after* A and C *after* B, then B will be *before* elements of both optical lines through C and *after* elements of both optical lines through A in any inertia plane containing a.

If a be an optical line and B be *after* A and C *after* B, then, *apart from a itself*, there is only one optical line through any element of a in any inertia plane containing a, and so we should have B *before* an element of the optical line through C and *after* an element of the parallel optical line through A.

If a be a separation line, however, we should have B *before* an element of one of the optical lines through C and *after* an element of the parallel optical line through A and also *after* an element of the second optical line through C and *before* an element of the parallel optical line through A.

The distinctions are perhaps exhibited more clearly by the following figures:



Inertia Line *Optical Line* *Separation Line*

Fig. 19.

From Theorem 70 it follows that the property of one element being linearly between two others is independent of the particular inertia plane in which the elements are considered as lying and so may be regarded as a relation of the one element to the other two.

This relation has been defined in terms of the relations *before* and *after*, not only for the cases where the three elements considered are such that of any two of them one is *after* the other; but also for the case of elements in a separation line when this is no longer so.

It is thus possible to state certain general results which hold for all three types of general line involving the conception linearly between.

Peano has given some eleven axioms of the *straight line* which are as follows:

(1) There is at least one point.

(2) If A is any point, there is a point distinct from A.

(3) If A is a point, there is no point lying between A and A.

(4) If A and B are distinct points, there is at least one point lying between A and B.

(5) If the point C lies between A and B, it also lies between B and A.

(6) The point A does not lie between the points A and B.

Definition. If A and B are points, the symbol AB represents the class of points such as C with the property that C lies between A and B.

Definition. If A and B are points, the symbol $A'B$ represents the class of points such as C with the property that B lies between A and C. Thus $A'B$ is the prolongation of the line beyond B, and $B'A$ its prolongation beyond A.

(7) If A and B are distinct points, there exists at least one member of $A'B$.

(8) If A and D are distinct points, and C is a member of AD and B of AC, then B is a member of AD.

(9) If A and D are distinct points, and B and C are members of AD, then either B is a member of AC, or B is identical with C, or B is a member of CD.

(10) If A and B are distinct points, and C and D are members of $A'B$, then either C is identical with D, or C is a member of BD, or D is a member of BC.

(11) If A, B, C, D are points, and B is a member of AC and C of BD, then C is a member of AD.

Definition. The straight line possessing A and B, symbolised by str. (A, B), is composed of the three classes $A'B$, AB, $B'A$ together with the points A and B themselves.

Of these axioms the writer has succeeded in proving nos. 6 and 9 from the others, so that they are really redundant.*

It is easy to see, with our definition of linearly between, that corresponding results hold for all three types of "general line".

As regards axioms (1) and (2) which we shall express thus:

(1) *There is at least one element,*

and (2) *If A be any element there is an element distinct from A,*

* *Messenger of Mathematics*, vol. XIII, pp. 121–123 and 134.

the first follows from our preliminary statement on p. 27, while the second follows directly from Posts. II and I and also from Post. V.

As regards axiom (3) we shall put it in the form:

(3) *If A is an element, there is no element lying linearly between A and A.*

This follows from the definition of linearly between.

(4) *If A and B are distinct elements, there is at least one element lying linearly between A and B.*

From our remarks at the end of Theorem 35 it appears that there are an infinite number of inertia planes containing any two distinct elements and accordingly any two distinct elements lie in a general line.

If A and B lie in an optical line, then Theorem 11 shows that there is at least one element which is *after* the one and *before* the other and is therefore linearly between them.

If A and B lie in an inertia line, the same result follows from Theorem 39; while if they lie in a separation line, it follows from Theorem 41.

(5) *If the element C lies linearly between A and B, it also lies linearly between B and A.*

This follows from the definition of linearly between.

(6) *The element A does not lie linearly between the elements A and B.*

This follows from the definition of what is meant by an element lying between a pair of parallel optical lines in an inertia plane. According to this definition the element must not lie in either optical line.

(7) *If A and B are distinct elements, there is at least one element such that B lies linearly between it and A.*

If A and B lie in an optical line or an inertia line, one of them must be *after* the other.

If it be the element A which is *after* B, then Theorems 7 and 38 show that there is at least one element of the general line which is *before* B, and so B lies linearly between it and A.

Similarly if A be *before* B there is an element of the general line which is *after* B, and so B is linearly between it and A.

If A and B lie in a separation line, the result follows from Theorem 43.

(8) *If A and D are distinct elements and C is linearly between A and D, and B linearly between A and C, then B is linearly between A and D.*

This is readily seen to be true if we take a set of parallel optical lines

a, b, c and d through A, B, C and D respectively in any inertia plane containing the four elements.

Let these optical lines intersect an optical line f of the opposite set in A', B', C' and D' respectively.

Remembering that Post. III must be satisfied, it is clear that we must have either:

(i) C' *after* D' and A' *after* C' together with B' *after* C' and A' *after* B';

or (ii) C' *before* D' and A' *before* C' together with B' *before* C' and A' *before* B'.

In case (i) it follows by Post. III that B' is *after* D' and consequently since B' is *before* A' we have B linearly between A and D.

Similarly in case (ii) we have B' *before* D' and *after* A', and therefore again B linearly between A and D.

(9) *If A and D are distinct elements and B and C are each linearly between A and D, then either B is linearly between A and C or B is identical with C or B is linearly between C and D.*

This result may be deduced in a similar manner to the last.

We must have either

(i) B' *after* D' and A' *after* B' together with C' *after* D' and A' *after* C';

or (ii) B' *before* D' and A' *before* B' together with C' *before* D' and A' *before* C'.

Then the elements B' and C' must either be identical or else the one is *after* the other.

In case (i) if B' be *after* C', since also B' is *before* A', we have B linearly between A and C.

If B' is identical with C', then B is identical with C.

If C' be *after* B', then since also D' is *before* B' we have B linearly between C and D.

Similarly in case (ii) we must either have B linearly between A and C or B identical with C or B linearly between C and D.

(10) *If A and B are distinct elements and if B is linearly between A and C and also linearly between A and D, then either C is identical with D, or C is linearly between B and D, or D is linearly between B and C.*

This result may also be deduced in a similar way. We must have either:

(i) B' *after* C' and A' *after* B' together with B' *after* D';

or (ii) B' *before* C' and A' *before* B' together with B' *before* D'.

Then the elements C' and D' must either be identical or else the one is *after* the other.

In case (i) if C' is *after* D', then since C' is *before* B' we have C linearly between B and D.

If C' is identical with D', then C is identical with D.

If D' is *after* C', then since D' is *before* B' we have D linearly between B and C.

Similarly in case (ii) we must either have C linearly between B and D or C identical with D, or D linearly between B and C.

(11) *If A, B, C, D are elements and B is linearly between A and C, and C is linearly between B and D, then C is linearly between A and D.*

This result may also be deduced in a similar way. We must have either:

(i) B' *after* C' and A' *after* B' together with C' *after* D';

or (ii) B' *before* C' and A' *before* B' together with C' *before* D'.

In case (i) since B' is *after* C' and A' *after* B', it follows by Post. III that A' is *after* C', and so C must be linearly between A and D.

Similarly in case (ii) we must also have C linearly between A and D.

Thus all these axioms of Peano hold for the general line.

Theorem 71

(*a*) *If A_0 and A_x be two elements in a general line a which lies in the same inertia plane with another general line b which intersects a in the element C such that either A_0 is linearly between C and A_x, or A_x is linearly between C and A_0, and if an optical line through A_0 intersects b in B_0 so that B_0 is* after *A_0, then a parallel optical line through A_x will intersect b in an element which is* after *A_x.*

We have already proved special cases of this in Theorems 66 and 67, and have now to prove the general theorem.

The optical line through A_x parallel to A_0B_0 must intersect b since b intersects A_0B_0 in B_0.

Let the element of intersection of this optical line through A_x with b be B_x.

Then B_x cannot be identical with A_x, for then the general lines a and b would have two distinct elements C and A_x in common and would therefore be identical, which is impossible since a and b intersect by hypothesis.

Further, if A_x were *after* B_x the general lines a and b would intersect in some element between the parallel optical lines (Theorem 68).

That is to say in some element linearly between A_0 and A_x.

But a and b have only one element C in common, so that if A_x were *after* B_x we should require C to be linearly between A_0 and A_x, contrary to the hypothesis that either A_0 is linearly between C and A_x or A_x is linearly between C and A_0.

Thus B_x must be *after* A_x.

(*b*) *If A_0 and A_x be two elements in a general line a which lies in the same inertia plane with another general line b which intersects a in the element C such that either A_0 is linearly between C and A_x or A_x is linearly between C and A_0, and if an optical line through A_0 intersects b in B_0 so that B_0 is* before *A_0, then a parallel optical line through A_x will intersect b in an element which is* before A_x.

Definition. We shall speak of a general line l as being *co-directional* with a general line m when l is either parallel to m or identical with it.

Theorem 72

If three parallel general lines a, b and c in one inertia plane P intersect a general line d_1 in A_1, B_1 and C_1 respectively and intersect a second general line d_2 in A_2, B_2 and C_2 respectively, then if B_1 is linearly between A_1 and C_1 we shall also have B_2 linearly between A_2 and C_2.

If a, b and c be optical lines, then we must either have b an after-parallel of c and a before-parallel of a, or else have b an after-parallel of a and a before-parallel of c.

In either case B_2 will be *after* an element of one of the pair of parallel optical lines a and c and *before* an element of the other.

Thus, as B_2 cannot lie in either a or c, and as these optical lines pass through A_2 and C_2 respectively, it follows that B_2 is linearly between A_2 and C_2.

Next consider the cases where a, b and c are separation or inertia lines: the methods of proof being similar in the two cases.

Let parallel optical lines in P and passing through A_1 and C_1 intersect b in B_1' and B_1'' respectively.

Let optical lines co-directional with these and passing through A_2 and C_2 intersect b in B_2' and B_2'' respectively.

Now as B_1 is supposed to be linearly between A_1 and C_1 it must lie between the parallel optical lines A_1B_1' and $B_1''C_1$.

It follows therefore, from Theorem 69, that if A_1 is *after* B_1' we shall also have B_1'' *after* C_1.

If, on the other hand, B_1' is *after* A_1 we shall also have C_1 *after* B_1''.

In the first of these cases, that is to say when A_1 is *after* B_1' and B_1'' *after* C_1, by Theorem 56 or Theorem 57 (according as a, b and c are separation or inertia lines), it follows that A_2 is *after* B_2' and B_2'' is *after* C_2.

If A_2B_2' is distinct from $B_2''C_2$ it follows, by Theorem 68, that B_2 must lie between the parallel optical lines A_2B_2' and $B_2''C_2$; so that B_2 is linearly between A_2 and C_2.

If A_2B_2' and $B_2''C_2$ are not distinct optical lines, then B_2' and B_2'' will both coincide with B, which will be *after* C_2 and *before* A_2; so that B_2 will still be linearly between A_2 and C_2.

The same result follows in a similar way in the case where B_1' is *after* A_1 and C_1 *after* B_1''.

Thus the theorem holds in all cases.

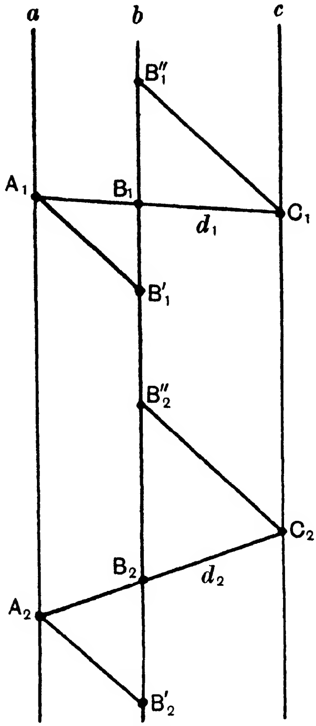


Fig. 20.

Theorem 73

(*a*) *If an element B be linearly between two elements A and C and if another element D be* before *both A and C but not in the general line AC, then DB is an inertia line and B is after D.*

Consider first the case where AC is a separation line.

Let a general line through B parallel to CD intersect AD in E.

Then since B is linearly between A and C we must have E linearly between D and A.

Thus since D is *before* A it follows that E is *after* D and *before* A.

But EB must be an inertia line or an optical line, according as DC is an inertia line or an optical line, and so B must be either *before* or *after* E.

But B cannot be *before* E for then, since E is *before* A, we should have B *before* A, contrary to the hypothesis that A and B lie in a separation line.

Thus B must be *after* E and, since E is *after* D, it follows that B is *after* D.

Thus DB is either an optical or an inertia line.

But if DB were an optical line, then since E is *after* D and *before* B it would follow that E must lie in DB, which is impossible since BE is parallel to CD.

Thus DB must be an inertia line.

Next consider the case where AC is an optical or inertia line.

We then must have either C *after* A or A *after* C and it is sufficient to consider the case where C is *after* A.

Then B must be *after* A and *before* C.

But A is *after* D and so B must be *after* D.

Thus again DB must be either an optical line or an inertia line.

If DB were an optical line, then since A is *after* D and *before* B the element A would have to lie in DB and so D, A and C would all lie in one general line, contrary to hypothesis.

Thus again DB must be an inertia line, and so the theorem is proved.

(*b*) *If an element B be linearly between two elements A and C and if another element D be* after *both A and C but not in the general line AC, then DB is an inertia line and B is* before *D.*

Remarks

A somewhat analogous result is the following:

If an element B be after *an element A and* before *an element C, and if D be another distinct element such that DA and DC are both separation lines, then DB is also a separation line.*

This may be proved as follows:

The element B cannot be *before* D; for, since A is *before* B, we should have A *before* D, contrary to the hypothesis that DA is a separation line.

Similarly B cannot be *after* D; for, since C is *after* B, we should have C *after* D, contrary to the hypothesis that DC is a separation line.

Thus B is neither *before* nor *after* D, so that DB must be a separation line, as was to be proved.

The following two theorems are special cases of Theorems 76 and 77, but as the proofs of the general theorems are reduced to depend on these special cases, the latter are treated separately.

Theorem 74

If A, B and C be three distinct elements in an inertia plane such that AB and AC are distinct optical lines, and if D be an element linearly between A and B while E is an element linearly between A and C, then there exists an element which lies both linearly between C and D and also linearly between B and E.

It will be sufficient to consider the case where A is *after* B, since the case where A is *before* B may be treated in an analogous manner.

Since E is linearly between A and C, therefore E is between the optical line AB and a parallel optical line through C.

Thus, A being supposed *after* B, this optical line through C will intersect the general line BE in some element, say G, such that G is *after* C (Theorem 69).

But since D is linearly between A and B we must, in these circumstances, have D *after* B.

Thus, since G is *after* C, it follows by Theorem 68 that the general lines BG and DC intersect in some element, say F, which is between the parallel optical lines DB and CG.

That is, F is linearly between C and D, and is the element of intersection of CD and BE.

By taking an optical line through B parallel to AC we may prove in an analogous manner that F is linearly between B and E.

Theorem 75

If A, B and C be three distinct elements in an inertia plane such that AB and AC are distinct optical lines and if D be an element linearly between A and B while F is an element linearly between C and D, then there exists an element, say E, which lies linearly between A and C and such that F lies linearly between B and E.

As in the previous theorem, it will be sufficient to consider only the case where A is *after* B.

Under these circumstances we should have D *after* B and so, since F is linearly between C and D, we should have F between the optical line AB and a parallel optical line through C.

Thus, by Theorem 69, this optical line through C will intersect BF in some element, say G, such that G is *after* C.

But, since A is *after* B and G is *after* C, therefore, by Theorem 68, the general lines AC and BG intersect in some element, say E, such that E is between the parallel optical lines AB and CG.

Thus E is linearly between A and C, where E is the element of intersection of AC and BF.

It then follows, as in the last theorem, that F lies linearly between B and E.

Theorem 76

If A, B and C be three elements in an inertia plane which do not all lie in one general line and if D be an element linearly between A and B, while E is an element linearly between A and C, there exists an element which lies both linearly between B and E and linearly between C and D.

Let V be the inertia plane containing A, B and C and let a be any inertia line through A which does not lie in V.

Let b and c be inertia lines parallel to a and passing through B and C respectively.

Then b and c lie in one inertia plane, say P_{bc}, c and a in a second inertia plane, say P_{ca}, and a and b in a third inertia plane, say P_{ab}.

Let one of the optical lines through B in the inertia plane P_{ab} intersect a in A' and let one of the optical lines through A' in the inertia plane P_{ca} intersect c in C'.

Then $A'B$ and $A'C'$ may be taken as generators of opposite sets of an inertia plane, say S, containing B, C' and A'.

Let d be the inertia line through D parallel to a and let e be the inertia line through E parallel to a.

Then d will lie in P_{ab} and, since D is linearly between A and B, it follows by Theorem 72 that d must intersect $A'B$ in some element, say D', such that D' is linearly between A' and B.

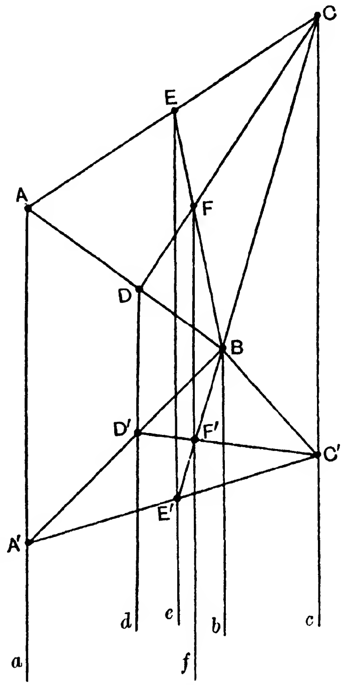


Fig. 21.

Similarly e will lie in P_{ca} and, since E is linearly between A and C, it follows that e will intersect $A'C'$ in some element, say E', such that E' is linearly between A' and C'.

But, since $A'B$ and $A'C'$ are two distinct optical lines in the inertia plane S, it follows by Theorem 74 that there exists an element, say

F', which lies both linearly between B and E' and linearly between C' and D'.

Now, since b and c are parallel inertia lines lying in the inertia plane P_{bc}, it follows that there is an inertia plane, say P_{bf}, containing b and the element F', and an inertia plane, say P_{cf}, containing c and the element F' and these inertia planes must, by Theorem 51, have a general line, say f, in common.

Further, f must be parallel to b and c and must therefore be an inertia line.

But the inertia plane P_{bf} contains the general line BF' and must therefore contain E' and the inertia line e which passes through E' and is parallel to b.

Thus P_{bf} contains the element E and therefore contains the general line BE.

Similarly P_{cf} contains the general line CD.

But, since F' is linearly between B and E', it follows by Theorem 72 that the inertia line f must intersect BE in some element, say F, such that F is linearly between B and E.

Similarly, since F' is linearly between C' and D', it follows that f must intersect CD in some element, say $\bar{F}$, such that $\bar{F}$ is linearly between C and D.

But both F and $\bar{F}$ must lie in V and so, if they were distinct, the inertia line f would require to lie in V.

But f is parallel to a, of which only one element lies in V and therefore f does not lie in V and accordingly $\bar{F}$ must be identical with F.

Thus the element F is both linearly between B and E and linearly between C and D.

It may happen in this and the next theorem that A' coincides with A or C' with C, but this does not affect the validity.

Theorem 77

If A, B and C be three elements in an inertia plane which do not all lie in one general line and if D be an element linearly between A and B while F is an element linearly between C and D, there exists an element, say E, which is linearly between A and C and such that F is linearly between B and E.

Let V be the inertia plane containing A, B and C and let a be any inertia line through A which does not lie in V, while b and c are inertia lines parallel to a through B and C respectively.

Let P_{bc}, P_{ca}, P_{ab}. A', C', S, d, D', have the same significance as in

the last theorem, and let P_{cd} be the inertia plane containing the parallel inertia lines c and d.

Let f be an inertia line through F parallel to c and d and which will also lie in P_{cd}.

Since F is linearly between C and D it follows, by Theorem 72, that f will intersect $C'D'$ in some element, say F', such that F' is linearly between C' and D'.

But, as in the last theorem, D' is linearly between A' and B and so, since $A'B$ and $A'C'$ are two distinct optical lines in the inertia plane S, it follows, by Theorem 75, that there exists an element, say E', which lies linearly between A' and C' and such that F' lies linearly between B and E'.

If now we denote the inertia plane containing b and f by P_{bf}, then P_{bf} contains the element E' in common with the inertia plane P_{ca}.

But, since b lies in P_{bc} and P_{bf} while the parallel inertia line c lies in P_{bc} and P_{ca}, it follows, by Theorem 51, that P_{bf} and P_{ca} have a general line, say e, in common which passes through E' and is parallel to b and c and is therefore an inertia line.

Now since a is also parallel to e and lies in the same inertia plane P_{ca} with it and, since E' is linearly between A' and C', it follows, by Theorem 72, that e must intersect AC in some element, say E, such that E is linearly between A and C.

Again, since b, f and e all lie in the inertia plane P_{bf} and, since F' is linearly between B and E', it follows, by Theorem 72, that BF must intersect e in some element $\bar{E}$ such that F is linearly between B and $\bar{E}$.

But both E and $\bar{E}$ must lie in V and so, if they were distinct, the inertia line e would require to lie in V.

But e is parallel to a, of which only one element lies in V and therefore e does not lie in V and accordingly $\bar{E}$ must be identical with E.

Thus the element E is linearly between A and C and is such that F is linearly between B and E.

Remarks

Peano has given the following three axioms of the plane:

(12) If r is a straight line, there exists a point which does not lie on r.

(13) If A, B, C are three non-collinear points and D lies on the segment BC, and E on the segment AD, there exists a point F on both the segment AC and the prolongation $B'E$.

(14) If A, B, C are three non-collinear points and D lies on the

segment BC and F on the segment AC, there exists a point E lying on both the segments AD and BF.

Now since there is always an element outside any general line it follows that the analogue of Peano's axiom (12) holds for our geometry.

Further, provided the three elements A, B, C lie in an inertia plane, Theorem 76 corresponds to Peano's axiom (14) while Theorem 77 corresponds to his axiom (13).

Also Theorem 47 corresponds to the axiom of parallels in Euclidean geometry so far as an inertia plane is concerned.

An inertia plane however differs from a Euclidean plane, since there are three types of general line in the former and only one type of straight line in the latter.

Further, although closed figures exist in an inertia plane, there is no closed figure which corresponds to a circle.

How this comes about will be seen hereafter.

THEOREM 78

If A, B and C be three elements in an inertia plane P which do not all lie in one general line and if D be the mean of A and B, then a general line through D parallel to BC intersects AC in an element which is the mean of A and C.

The general line BC, which we shall denote by a, may be either: (i) an optical line, (ii) a separation line, or (iii) an inertia line.

As regards case (i): let a' be an optical line through A parallel to a, and let AK be the second optical line which passes through A and lies in the inertia plane P, and let it intersect a in the element K.

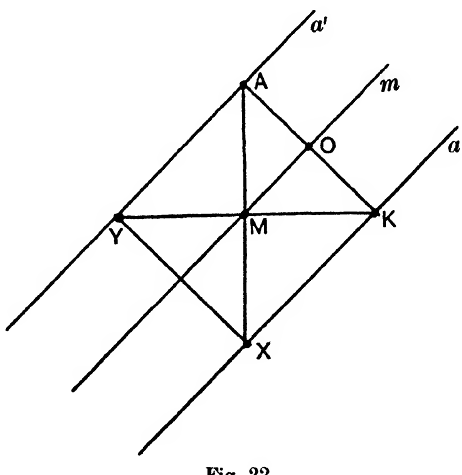


Fig. 22.

Let X be any other element in a and let the optical line through X parallel to KA intersect a' in Y. Then A, K, X, Y are the corners of an optical parallelogram whose diagonal lines AX and YK intersect in some element, say M, which is the mean both of A and X and of Y and K.

An optical line m through M parallel to a will intersect AK in an element (say O) which is the mean of A and K.

The position of O is independent of the position of X in the optical line a, so that if B and C be any two elements in a, and D be the mean of A and B the general line through D parallel to BC will be identical with m and will intersect AC in an element which is the mean of A and C.

Case (ii) may be proved as follows:

Let any inertia line in P which passes through A intersect a in some element K, and let X be any other element in a.

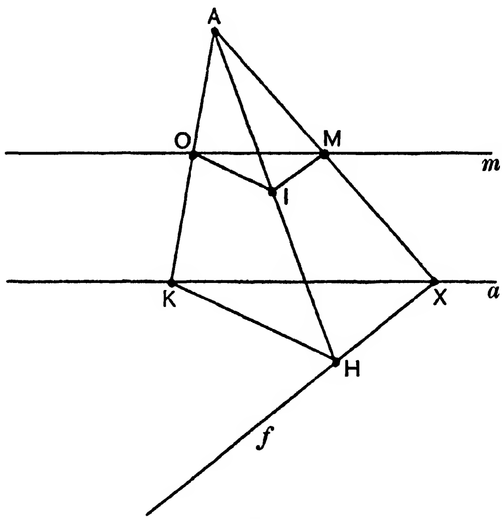


Fig. 23.

The element A will be either *before* or *after* K, but the method of proof is analogous in both cases and so we shall consider the one where A is *after* K.

Now, since a is a separation line, K will be neither *before* nor *after* X.

If then Q be any inertia plane containing a but distinct from P, the element K will be *before* an element of one of the optical lines in Q which pass through X and *after* an element of the other.

Let f be the optical line through X in the inertia plane Q such that

K is *after* an element of it and let the optical line through K in Q of the opposite set to f intersect f in H. Then K is *after* H.

But since A is supposed to be *after* K we shall have A *after* H and AH will be an inertia line.

Thus A, H and X will lie in an inertia plane and A, H and K will lie in another inertia plane.

Let O be the mean of A and K.

Then, by case (i), an optical line through O parallel to KH will intersect AH in some element I such that I is the mean of A and H.

Also, by case (i), an optical line through I parallel to HX will intersect AX in some element M which will be the mean of A and X.

But now I does not lie in Q and so O, I and M lie in an inertia plane, say Q', which will be parallel to Q.

Thus, since P has the separation line a in common with Q and has the general line OM (which we shall denote by m) in common with the parallel inertia plane Q', it follows that m is a separation line parallel to a and intersecting AX in an element M which is the mean of A and X.

For all positions of X the mean of A and X lies on the separation line m which passes through O which is the mean of A and K.

Thus if B and C be any two elements in the separation line a and D be the mean of A and B, then D will lie in the separation line m, which will also pass through the mean of A and C and is parallel to BC.

Case (iii) may be proved as follows:

Let any inertia line in P which passes through A and is not parallel to a intersect a in some element K and let X be any other element in a.

Let R be any inertia plane distinct from P but containing the general line AX.

Now there are two optical lines in R which pass through X and one at least of these optical lines must be distinct from AX. Let l be such an optical line.

Now, since a is an inertia line, it follows that a and l lie in an inertia plane which we shall denote by Q.

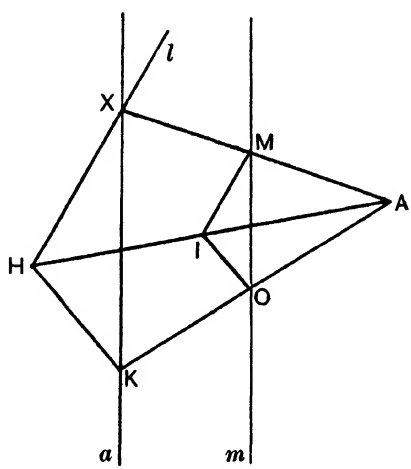


Fig. 24.

Let the optical line through K in the inertia plane Q of the opposite set to l intersect l in the element H.

Then, since AK is an inertia line, it follows that the three elements H, A and K lie in an inertia plane.

Let O be the mean of A and K.

Then, by case (i), an optical line through O parallel to KH will intersect AH in some element I such that I is the mean of A and H.

Also, by case (i), an optical line through I parallel to HX will intersect AX in some element M which will be the mean of A and X.

But now I does not lie in Q and so O, I and M lie in an inertia plane, say Q', which will be parallel to Q.

Thus, since P has the inertia line a in common with Q and has the general line OM (which we shall denote by m) in common with the parallel inertia plane Q', it follows that m is an inertia line parallel to a and intersecting AX in an element M which is the mean of A and X.

For all positions of X the mean of A and X lies on the inertia line m which passes through O which is the mean of A and K.

If then B and C be any two elements in the inertia line a and D be the mean of A and B, it will lie in the inertia line m, which will also pass through the mean of A and C and is parallel to BC.

Thus in all cases the theorem holds.

Since there is only one general line through D parallel to BC and this must pass through the mean of A and C, it follows directly that, if E be the mean of A and C, then the general line DE is parallel to BC.

Definition. If two parallel general lines in an inertia plane be both intersected by another pair of parallel general lines, then the four general lines will be said to form a *general parallelogram in the inertia plane*.

It will be seen hereafter that it is necessary to extend the meaning of the phrase *general parallelogram* to the case of figures which do not lie in an inertia plane and so the words "*in an inertia plane*" are important.

The general lines which form a general parallelogram in an inertia plane will be called the *side lines* of the general parallelogram.

A pair of parallel side lines will be said to be *opposite*.

The elements of intersection of pairs of side lines which are not parallel will be called the *corners* of the general parallelogram.

A pair of corners which do not lie in the same side line will be said to be *opposite*.

A general line passing through a pair of opposite corners will be called a *diagonal line* of the general parallelogram.

It is clear that a general parallelogram in an inertia plane has two diagonal lines.

Further, it is clear that an optical parallelogram is a particular case of a general parallelogram in an inertia plane.

Theorem 79

If we have a general parallelogram in an inertia plane, then:

(1) *The two diagonal lines intersect in an element which is the mean of either pair of opposite corners.*

(2) *A general line through the element of intersection of the diagonal lines and parallel to either pair of side lines, intersects either of the other side lines in an element which is the mean of the pair of corners through which that side line passes.*

Let A, B, C, D be the corners of the general parallelogram:

A and C being one pair of opposite corners and B and D the other pair.

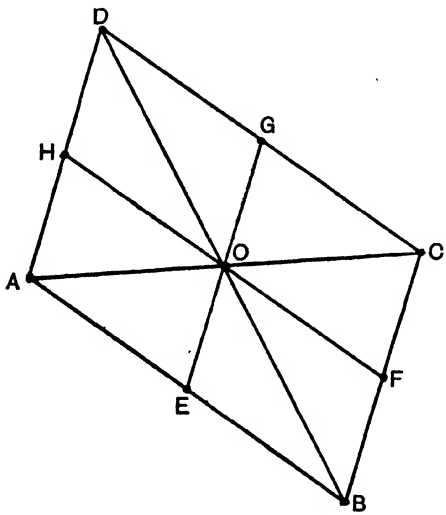


Fig. 25.

Let E be the mean of A and B,
,, F ,, B ,, C,
,, G ,, C ,, D,
,, H ,, D ,, A.

If a general line be taken through E parallel to BC and AD, then by Theorem 78 it will intersect AC in an element which is the mean of A and C and therefore will intersect CD in an element which is the mean of C and D. That is in the element G.

Similarly the general line FH will pass through the mean of A and C.

Thus the element of intersection of EG and FH (which we shall call O) is the mean of A and C. Similarly O is the mean of B and D. Thus the mean of A and C is identical with the mean of B and D; or the diagonal lines intersect in an element which is the mean of either pair of opposite corners. The second part of the theorem also holds.

Theorem 80

If A, B, C, D be the corners of a general parallelogram in an inertia plane; AB and DC being one pair of parallel side lines and BC and AD the other pair of parallel side lines, then if E be the mean of A and B while F is the mean of D and C, the general lines AF and EC are parallel to one another.

Since the general line AF is not parallel to BC, it must intersect BC in some element, say G.

Now by Theorem 79 a general line through the intersection of the diagonal lines and parallel to BC will intersect AB in the mean of A and B, and will intersect DC in the mean of D and C.

Thus the general line EF is parallel to BC.

But since A, B and G are three elements in an inertia plane which do not all lie in one general line and since E is the mean of A and B while EF is parallel to BG, it follows by Theorem 78 that F is the mean of A and G.

Similarly since FC is parallel to AB it follows that C is the mean of G and B.

But since E is the mean of B and A while C is the mean of B and G, it follows by Theorem 78 that EC is parallel to AG: that is, EC is parallel to AF, as was to be proved.

Theorem 81

If three parallel general lines a, b and c in one inertia plane intersect a general line d_1 in A_1, B_1 and C_1 respectively and intersect a second general line d_2 in A_2, B_2 and C_2 respectively, and if B_1 be the mean of A_1 and C_1, then B_2 will be the mean of A_2 and C_2.

If A_2 should happen to coincide with A_1, or if C_2 should happen to coincide with C_1, the result follows directly from Theorem 78.

If d_2 should happen to be parallel to d_1, then the result follows from Theorem 79 (2).

In any other case let a general line through A_1 parallel to d_2 intersect b in B and c in C.

Then, by Theorem 78, B is the mean of A_1 and C and so, by Theorem 79 (2), B_2 will be the mean of A_2 and C_2.

Remarks

If A_0 and A_n be two distinct elements in a general line a, we can always find $n-1$ elements $A_1, A_2, \dots A_{n-1}$ in a (where $n-1$ is any integer) such that:

A_1 is the mean of A_0 and A_2,
A_2 is the mean of A_1 and A_3,
....
....
A_{n-1} is the mean of A_{n-2} and A_n.

For let P be any inertia plane containing a and let b be any general line in P which passes through A_0 and is distinct from a.

Let A_1' be any element in b distinct from A_0 and let $A_2', A_3', \dots A'_{n-1}$, A_n' be other elements in b such that:

A_1' is the mean of A_0 and A_2',
A_2' is the mean of A_1' and A_3',
....
....
A'_{n-1} is the mean of A'_{n-2} and A_n'.

Let general lines through $A_1', A_2', \dots A'_{n-1}$ parallel to $A_n'A_n$ intersect a in the elements $A_1, A_2, \dots A_{n-1}$.

Then, by Theorem 81, it follows that:

A_1 is the mean of A_0 and A_2,
A_2 is the mean of A_1 and A_3,
....
....
A_{n-1} is the mean of A_{n-2} and A_n,

and so the $n-1$ elements $A_1, A_2, \dots A_{n-1}$ can be found as stated.

Theorem 82

(*a*) *If A be any element in an optical line a and A' be any element in a neutral-parallel optical line a', then, if B be any element in a which is* after *A, the general line through B parallel to AA' intersects a' in an element which is* after *A'.*

Since A and A' lie in the neutral-parallel optical lines a and a' respectively, it follows that A is neither *before* nor *after* A' and so there is at least one element which is common to the α sub-sets of A and A'.

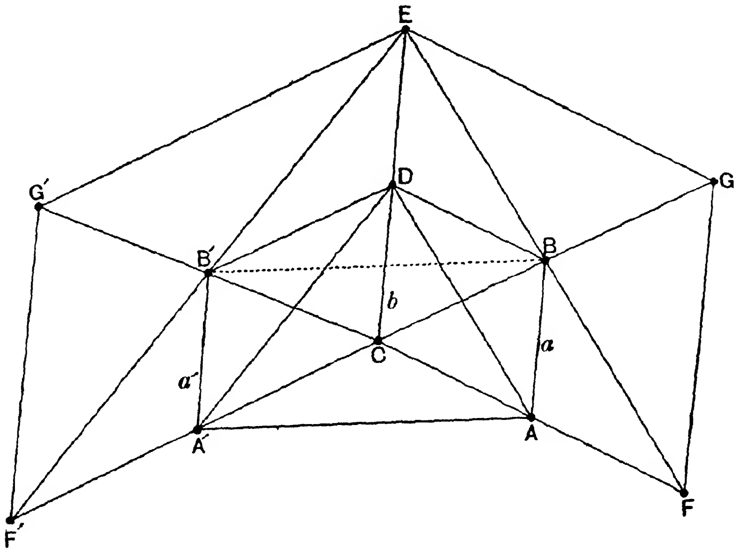


Fig. 26.

Let C be such an element and let b be the optical line through C parallel to a or a'.

Then since C is *after* both A and A', it follows that b is an after-parallel of both a and a' and accordingly b and a lie in one inertia plane while b and a' lie in another.

Let the optical line through B parallel to AC intersect b in the element D and let the optical line through D parallel to CA' intersect a' in the element B'.

Then A, C, D, B form the corners of an optical parallelogram in an inertia plane which we shall call P, while A', C, D, B' form the corners of another optical parallelogram in another inertia plane which we shall call P'.

Now, since B is *after* A and C is also *after* A, while AC and AB are both optical lines, it follows that the diagonal line CB is a separation

line and accordingly the diagonal line AD is an inertia line having D *after* A.

Further, D must be *after* C and, since C is *after* A', it follows that the diagonal line $A'D$ is an inertia line having D *after* A', and accordingly the diagonal line CB' is a separation line.

Thus, since C is *after* A', we must also have B' *after* A'.

Let the general line through B parallel to AD intersect b in E and CA in F, and let the optical lines through E and F respectively parallel to CF and CE intersect one another in G.

Then F, C, E, G are the corners of an optical parallelogram in the same inertia plane as the optical parallelogram whose corners are A, C, D, B and the diagonal lines FE and AD do not intersect and so the diagonal lines CG and CB do not intersect.

Thus B must lie in CG and since it also lies in FE it follows that B is the centre of the optical parallelogram whose corners are F, C, E, G.

Now let the general line through E parallel to DA' intersect CA' in F' and let the optical lines through E and F' respectively parallel to CF' and CE intersect one another in G'.

Then F', C, E, G' are the corners of an optical parallelogram in the same inertia plane as the optical parallelogram whose corners are A', C, D, B' and the diagonal lines $F'E$ and $A'D$ do not intersect and so the diagonal lines CG' and CB' do not intersect.

Thus B' lies in CG'.

But the optical parallelograms whose corners are F, C, E, G and F', C, E, G' have the pair of adjacent corners C and E in common and the optical line BD through the centre of the first of these intersects CE in D, and so it follows by Theorem 61 that the centre of the second optical parallelogram lies in the optical line through D parallel to CF' and EG'.

Thus the centre of the optical parallelogram whose corners are F', C, E, G' lies in DB'.

But this centre also lies in CG' and therefore it must be the element B'.

Thus B' must lie in $F'E$.

But we saw that AD and $A'D$ were both inertia lines and so they lie in an inertia plane, say Q_1, while BE and $B'E$ which are respectively parallel to these must lie in a parallel inertia plane, say Q_2.

Further, AC and $A'C$ are both optical lines and so they lie in an inertia plane, say R_1, while BD and $B'D$ which are respectively parallel to these must lie in a parallel inertia plane, say R_2.

But the general lines AA' and BB' lie in the parallel inertia planes

Q_1 and Q_2 respectively and also in the parallel inertia planes R_1 and R_2 respectively, and since these inertia planes are distinct it follows that BB' is parallel to AA'.

Thus the parallel to AA' through B intersects a' in the element B' which is *after* A'.

(*b*) *If A be any element in an optical line a and A' be any element in a neutral-parallel optical line a', then if B be any element in a which is* before A, *the general line through B parallel to AA' intersects a' in an element which is* before A'.

Theorem 83

If A and B be two elements lying respectively in the two neutral-parallel optical lines a and b, and if A' be a second and distinct element in a, there is only one general line through A' and intersecting b which does not intersect the general line AB.

We have seen by Theorem 82 (a and b) that the general line through A' parallel to AB must intersect b.

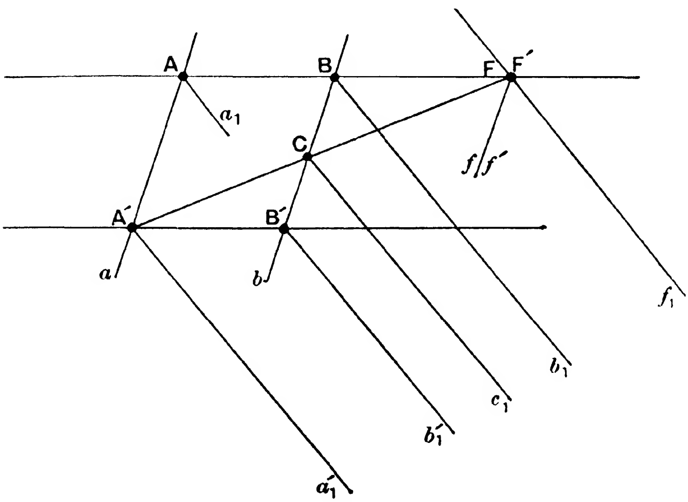


Fig. 27.

Let B' be the element of intersection.

Then the general lines AB and $A'B'$, being parallel, cannot intersect.

Let any other general line through A' and intersecting b intersect it in the element C.

Then if C should coincide with B the general lines $A'C$ and AB have the element B in common and therefore intersect.

Suppose next that C does not coincide with B.

Let P_1 be any inertia plane containing a and let P_2 be the parallel inertia plane containing b.

Let a_1 be any inertia line through A in the inertia plane P_1 and let Q be the inertia plane containing a_1 and the element B.

Then Q must contain a general line, say b_1, in common with P_2 and the general lines a_1 and b_1 must be parallel.

Again let a_1' be a general line through A' parallel to a_1.

Then a_1' must lie in the inertia plane P_1 and must be an inertia line.

Thus the general line a_1' and the element B' must lie in an inertia plane, say Q', and since a_1' is parallel to a_1 and $A'B'$ is parallel to AB, it follows by Theorem 52 that Q' is parallel to Q.

But the inertia plane Q' contains the general line a_1' in P_1 and the element B' in P_2 and therefore since P_1 and P_2 are parallel it follows that Q' and P_2 contain a general line, say b_1', in common, which will be parallel to a_1'.

Again, since a_1' is an inertia line, there is an inertia plane containing a_1' and the element C.

If we call this inertia plane R, then by Theorem 51 the inertia planes P_2 and R have a general line, say c_1, in common and c_1 is parallel to a_1' and b_1'.

Thus since c_1 lies in P_2 and R, b_1' in Q' and P_2, and a_1' in R and Q', and since Q is an inertia plane parallel to Q' through the element B of P_2 which does not lie in b_1', it follows by Theorem 53 that the inertia planes R and Q have a general line in common, say f_1, which is parallel to a_1'.

Now since C is neither *before* nor *after* A', it follows that $A'C$ is a separation line and therefore must intersect the inertia line f_1 since both lie in one inertia plane R.

Similarly AB is a separation line and must intersect the inertia line f_1 since both lie in the inertia plane Q.

Let AB intersect f_1 in F and let $A'C$ intersect f_1 in F'.

We have to show that F' is identical with F.

Let f be the optical line through F parallel to a and let f' be the optical line through F' parallel to a.

Then since B is neither *before* nor *after* any element of a, it follows by Theorem 45 that no element of the general line AB with the exception of A is either *before* or *after* any element of a; and similarly no element of the general line $A'C$ with the exception of A' is either *before* or *after* any element of a.

But F cannot be identical with A, for this would require C to lie in

P_1, which is impossible, and F' cannot be identical with A' since F' and A' lie in parallel inertia planes Q and Q'.

Thus F is neither *before* nor *after* any element of a and F' is neither *before* nor *after* any element of a.

It follows that f is a neutral-parallel of a and also f' is a neutral-parallel of a.

Suppose now, if possible, that F' is distinct from F; then since F and F' lie in the inertia line f_1, it would follow that the one was *after* the other.

Also if they were distinct, since they both lie in the same inertia line they could not also lie in one optical line and so the optical lines f and f' would be distinct and the one would be an after-parallel of the other.

But we have seen that f and f' are each neutral-parallels of a and so it would follow by Theorem 28 that they were neutrally parallel to one another.

But one optical line cannot be both a neutral-parallel and an after-parallel of another optical line and so the supposition that F' is distinct from F leads to a contradiction and therefore is not true.

Thus F' is identical with F and therefore the general line $A'C$ intersects the general line AB.

Thus there is no general line through A' and intersecting b which does not also intersect AB, except the parallel general line $A'B'$.

Theorem 84

If a and b be two neutral-parallel optical lines and if one general line intersects a in A and b in B, while a second general line intersects a in A' and b in B', then an optical line through any element of AB and parallel to a or b intersects $A'B'$.

Let D be any element of AB and let d be an optical line through D parallel to a or b.

We have to show that d intersects $A'B'$.

If D should coincide with either A or B, no proof is required.

If $A'B'$ be parallel to AB, then the result follows directly by Theorem 82 (a and b).

If $A'B'$ be not parallel to AB, then by Theorem 83 the general lines AB and $A'B'$ must intersect in some element, say C.

Now, the general lines AB and $A'B'$ being supposed distinct, C must be distinct from at least one of the elements A and B and without limitation of generality we may suppose that C is distinct from B.

Let Q be any inertia plane containing the optical line b and let b_1 be any inertia line through B in Q.

Let b_1' be the parallel inertia line through B' which will also lie in Q.

Let P be the inertia plane containing b_1 and C, while R is the inertia plane containing b_1' and C.

Then by Theorem 51 P and R have a general line, say c_1, in common, which is parallel to b_1 and b_1'.

Suppose that D is not identical with B and let Q' be the inertia plane through D and parallel to Q.

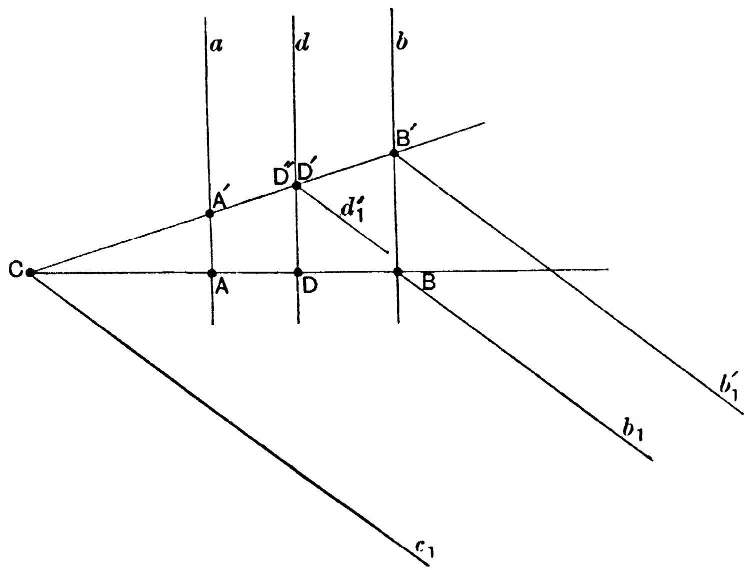


Fig. 28.

Then we have the three distinct inertia planes P, Q and R and the three parallel general lines c_1, b_1 and b_1', such that c_1 lies in P and R, b_1 in Q and P, and b_1' in R and Q, while Q' is an inertia plane parallel to Q through an element of P which does not lie in b_1, and so by Theorem 53 the inertia planes R and Q' have a general line in common which is parallel to b_1'.

Call this general line d_1'.

Then d_1' is an inertia line.

Now the optical line d must lie in Q' and must therefore intersect d_1' in some element, say D'.

Also $A'B'$ being a separation line in the inertia plane R must intersect the inertia line d_1' in some element, say D''.

We have to show that D'' is identical with D'.

Suppose if possible that D'' is distinct from D' and let d'' be the optical line through D'' parallel to b.

Then since by Theorem 45 D is neither *before* nor *after* any element of b, it follows that d is a neutral-parallel of b.

Similarly d'' is a nuetral-parallel of b and so if D' and D'' were distinct and did not lie in the same optical line, it would follow by Theorem 28 that d'' was a neutral-parallel of d.

But D' and D'' lie in d_1', which is an inertia line, and so if D' and D'' were distinct one of them would have to be *after* the other and so d and d'' could not be neutral-parallels.

Thus the supposition that D'' is distinct from D' leads to a contradiction and so D'' must be identical with D'.

Thus the optical line d intersects $A'B'$ in D', which proves the theorem.

Theorem 85

If a and b be two neutral-parallel optical lines and E be any element in a separation line AB which intersects a in A and b in B, and if $A'B'$ be any other separation line intersecting a in A' and b in B' but not parallel to AB, then E either lies in $A'B'$ or in a separation line parallel to $A'B'$ which intersects both a and b.

If E does not lie in $A'B'$, then by Theorem 84 an optical line through E parallel to a or b intersects $A'B'$ in some element, say E', which is either *before* or *after* E.

Thus by Theorem 82 the general line through E parallel to $A'B'$ intersects a and similarly it intersects b.

Thus E must lie in a separation line parallel to $A'B'$ and intersecting both a and b when it does not lie in $A'B'$ itself.

Remarks

If a and b be two neutral-parallel optical lines and if c and d be any two non-parallel separation lines intersecting both a and b, then it is evident from Theorem 85 that: the aggregate consisting of all the elements in c and in all separation lines intersecting a and b which are parallel to c must be identical with the aggregate consisting of all the elements in d and in all separation lines intersecting a and b which are parallel to d.

This follows since each element in the one set of separation lines must also lie in the other set.

Thus the aggregate which we obtain in this way is independent of

the particular set of parallel separation lines intersecting a and b which we may select and so we have the following definition.

Definition. The aggregate of all elements of all mutually parallel separation lines which intersect two neutral-parallel optical lines will be called an *optical plane.**

It is evident that through any element of an optical plane there is *one single optical line* lying in the optical plane.

For if a and b be two neutral-parallel optical lines which are intersected by a separation line d in the elements A and B respectively, and if C be any other element in d, then there is a neutral-parallel to a and b through C which we may call c.

But through each element of c other than C there is a separation line parallel to d which, by Theorem 82 (a and b), must intersect both a and b, and so every element of the optical line c lies in the optical plane defined by a and b.

An optical plane differs in this respect from an inertia plane, since the latter contains two optical lines passing through any element of it.

Definition. In analogy with the case of an inertia plane, an optical line which lies in any optical plane will be called a *generator* of the optical plane.

THEOREM 86

If two distinct elements of a general line lie in an optical plane, then every element of the general line lies in the optical plane.

Let the optical plane be determined by the two neutral-parallel optical lines a and b.

If the two elements lie in a general line which is known to intersect both a and b, no proof is required.

Let C be any element in any separation line AB which intersects a in A and b in B, and let D' be any element in any separation line $A'B'$ parallel to AB and intersecting a in A' and b in B'.

We have to show that every element of the general line CD' lies in the optical plane.

By Theorem 82 (a or b) an optical line through C parallel to a or b will intersect $A'B'$ in some element, say C'.

If C' should coincide with D', then CD' would be an optical line which would be neutrally parallel to a or b and we already know that

* The name "optical plane" has been adopted because of certain analogies with an optical line.

each element of it must lie either in a separation line parallel to AB and intersecting both a and b, or in AB itself.

Thus if C' should coincide with D', the general line CD' is such that every element of it lies in the optical plane.

If C' does not coincide with D', then an optical line through D' parallel to CC' will intersect AB in some element, say D (Theorem 82 (a or b)).

Now DD' must be a neutral-parallel of CC' and either of the optical lines a or b must be either parallel to CC' and DD' or identical with one of them.

If a is identical with CC' or DD', then a intersects CD', while if b is identical with CC' or DD', then b intersects CD'.

If a is not identical with CC' or DD', then, by Theorem 84, a must intersect CD', and similarly if b is not identical with CC' or DD', then b must intersect CD'.

Thus in all these cases CD' intersects both a and b and therefore every element of CD' lies in the optical plane determined by a and b.

THEOREM 87

If e be a general line in an optical plane and A be any element of the optical plane which does not lie in e, then there is one single general line through A in the optical plane which does not intersect e.

We saw in the course of proving Theorem 86 that if an optical plane be determined by two neutral-parallel optical lines a and b, then any general line containing two elements in the optical plane and therefore any general line lying in the optical plane, must either be a neutral-parallel of a or b, or else must intersect both a and b.

Suppose first that e is a separation line in the optical plane determined by a and b, then e must intersect both a and b.

Since A does not lie in e it must lie in a separation line f parallel to e and intersecting both a and b.

Now through A there is an optical line, say c, which is a neutral-parallel of a or b and which by Theorem 82 (a and b) must intersect e and must lie in the optical plane, while any other general line f' through A and lying in the optical plane must intersect both a and b.

But f' is not parallel to e and therefore by Theorem 83 it must intersect it.

Suppose next that e is an optical line.

Then e must either be parallel to a and b or be identical with one of them.

Through A there is an optical line parallel to a or b and therefore parallel to e, and this optical line must lie in the optical plane.

Any other general line through A in the optical plane intersects both a and b and so by Theorem 84 it must also intersect e.

Thus there is in all cases one single general line through A in the optical plane which does not intersect e.

Theorem 88

If A, B and C be three elements in an optical plane which do not all lie in one general line and if D be an element linearly between A and B, while E is an element linearly between A and C, there exists an element which lies both linearly between B and E and linearly between C and D.

The proof of this theorem is quite analogous to that of Theorem 76, the only difference being that V is here an optical plane instead of an inertia plane and, as such, it cannot contain any inertia line.

Thus the words "which does not lie in V" may be omitted from the first sentence of the proof.

Theorem 89

If A, B and C be three elements in an optical plane which do not all lie in one general line and if D be an element linearly between A and B while F is an element linearly between C and D, there exists an element, say E, which is linearly between A and C and such that F is linearly between B and E.

The proof of this theorem is quite analogous to that of Theorem 77, the only difference being that V is here an optical plane instead of an inertia plane and, as such, it cannot contain any inertia line.

Thus the words "which does not lie in V" may be omitted from the first sentence of the proof.

Remarks

It will be observed that Theorem 88 is the analogue of Peano's axiom (14) for the case of elements in an optical plane, while Theorem 89 is the corresponding analogue of his axiom (13).

Further, Theorem 87 corresponds to the Euclidean axiom of parallels for the case of general lines in an optical plane.

THEOREM 90

If A, B and C be three elements in an optical plane which do not all lie in one general line and if D be an element linearly between A and B while DE is a general line through D parallel to BC and intersecting AC in the element E, then E is linearly between A and C.

In the first place E cannot be identical with A for then the general line DE would be identical with the general line BA and would therefore intersect BC.

Again E cannot be identical with C for once more BC and DE would intersect.

Thus we must either have C linearly between A and E, or A linearly between C and E, or E linearly between A and C.

If C were linearly between A and E, then since D is linearly between A and B it would follow by Theorem 88 that there existed an element which was both linearly between B and C and linearly between E and D.

Thus in this case also BC and DE would intersect.

Next if A were linearly between C and E, then since D is linearly between A and B it would follow similarly by Theorem 89 that BC and DE must intersect.

Thus the only possibility is that E is linearly between A and C.

THEOREM 91

If three parallel general lines a, b and c in one optical plane intersect a general line d_1 in A_1, B_1 and C_1 respectively and intersect a second general line d_2 in A_2, B_2 and C_2 respectively, then if B_1 is linearly between A_1 and C_1 we shall also have B_2 linearly between A_2 and C_2.

If A_1 should be identical with A_2 the result follows directly from Theorem 90.

Similarly it follows directly if C_1 should be identical with C_2.

If B_1 should be identical with B_2 the following method is still valid.

The general line C_1A_2 cannot be identical with the general line c and therefore C_1A_2 must intersect the general line b (which is parallel to c) in some element, say B'.

Then, since B_1 is linearly between A_1 and C_1, it follows by Theorem 90 that B' is linearly between C_1 and A_2.

Similarly, since B' is linearly between A_2 and C_1, it follows that B_2 is linearly between A_2 and C_2.

THEOREM 92

If two elements A and B lie in one optical line and if two other elements C and D lie in a neutral-parallel optical line, and if A be after *B, then:*

(1) *If C be* after *D the general lines AD and BC intersect in an element which is both linearly between A and D and linearly between B and C.*

(2) *If the general lines AD and BC intersect in an element E which is either linearly between A and D or linearly between B and C, we shall also have C* after *D.*

Let A and B lie in an optical line a and let C and D lie in a neutral-parallel optical line c.

Let a_1 be any inertia line through A and let b_1 be a parallel inertia line through B.

Then a_1 and b_1 lie in an inertia plane, say P.

Let B' be any element in b_1 which is *after* B and let a' be an optical line through B' parallel to a.

Then a' will intersect a_1 in some element, say A', and, by Theorem 57, since A is *after* B, we must have A' *after* B'.

But, since B' is not an element of a but is *after* B, an element of a, it follows that a' is an after-parallel of a.

Since further a and c are neutral-parallels, it follows by Theorem 26 (b) that a' is an after-parallel of c.

Thus a' and c lie in an inertia plane, say Q.

Proceeding now to prove the first part of the theorem, we have A' *after* B' and C *after* D and so it follows by Theorem 68 and the definition of "linearly between" that $A'D$ and $B'C$ intersect in an element, say E', which is linearly between A' and D and also linearly between B' and C.

But since a_1 is an inertia line there is an inertia plane containing a_1 and the element E' which we may call R, and similarly there is an inertia plane containing b_1 and the element E' which we may call S.

Now since a_1 and b_1 are parallel general lines in the inertia plane P it follows, by Theorem 51, that the inertia planes R and S have a general line, say e_1, in common, which is parallel to a_1 and b_1 and must therefore be an inertia line.

Since e_1 lies both in R and S it must intersect BC and AD which lie respectively in S and R and are separation lines.

Suppose e_1 intersects BC in E and AD in $\bar{E}$, then E and $\bar{E}$ lie in the inertia line e_1 and so, if they were distinct, they could not lie in one optical plane.

But E and $\bar{E}$ each lie in the optical plane determined by the neutral-parallel optical lines a and c and so $\bar{E}$ is identical with E.

But since D, A and A' are elements in the inertia plane R which do not all lie in one general line, and since E' is linearly between A' and D

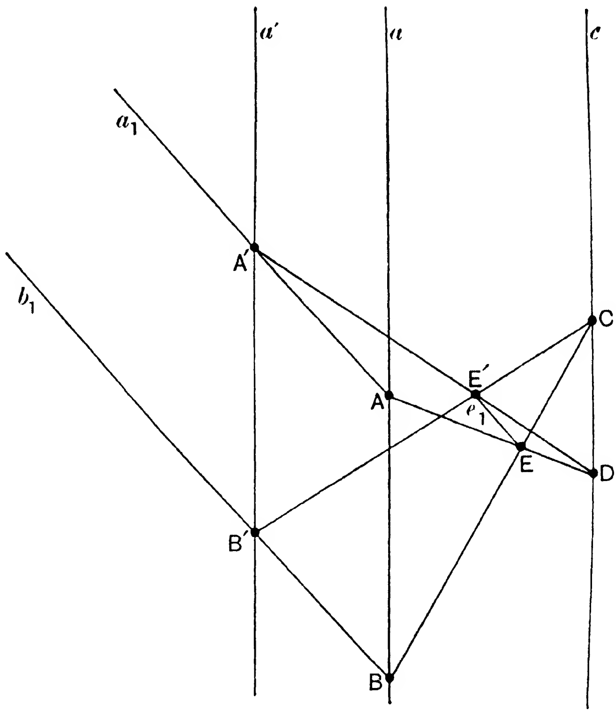


Fig. 29.

and $E'E$ is parallel to $A'A$, it follows, by Theorem 72, that E is linearly between A and D.

Similarly since E' is linearly between B' and C, and C, B and B' lie in the inertia plane S and are not all in one general line and since $E'E$ is parallel to $B'B$ it follows that E is linearly between B and C.

Thus the first part of the theorem is proved.

Proceeding now to prove the second part of the theorem; since AD and BC intersect in the element E and since a_1 and b_1 are inertia lines it follows that there is an inertia plane, say R, which contains a_1 and the element E, and another inertia plane, say S, containing b_1 and the element E.

It follows, since a_1 and b_1 are parallel and lie in the inertia plane P, that R and S have a general line, say e_1, in common, which is parallel to a_1 and b_1 (Theorem 51) and must therefore be an inertia line.

Now the element E could not lie in the optical line c, since then it would have to coincide with both C and D and could not therefore be linearly between A and D or linearly between B and C.

Thus, since E and c lie in one optical plane and c also lies in the inertia plane Q, it follows that E does not lie in Q and so the inertia line e_1 cannot have more than one element in common with Q.

If now E be linearly between A and D, then since D, A and A' lie in the inertia plane R and are not all in one general line, it follows, since e_1 is parallel to AA', that e_1 must intersect $A'D$ in an element, say E', which is linearly between A' and D.

Also, since $B'C$ is not parallel to e_1 and lies in the inertia plane S with it, it follows that $B'C$ must intersect e_1.

But $B'C$ lies in Q and we have seen that e_1 and Q cannot have more than one element in common and therefore $A'D$ and $B'C$ intersect e_1 in the same element E'.

If we suppose instead that E is linearly between B and C, we find in a similar way that $B'C$ and $A'D$ intersect e_1 in an element E' which is linearly between B' and C.

But by the definition of "linearly between" the element E' must in either case be between the parallel optical lines a' and c in the inertia plane Q.

Thus, since a' and c are parallel optical lines in the inertia plane Q and since A' is *after* B' and the element of intersection of $A'D$ and $B'C$ lies between a' and c, it follows by Theorem 69 that C is *after* D, as was to be proved.

Theorem 93

(*a*) *If A_0 and A_x be two elements in a general line a which lies in the same optical plane with another general line b which intersects a in the element C such that either A_0 is linearly between C and A_x, or A_x is linearly between C and A_0 and if an optical line through A_0 intersects b in B_0 so that B_0 is* after *A_0, then a parallel optical line through A_x will intersect b in an element which is* after *A_x.*

The proof of this theorem is exactly analogous to that of Theorem 71, using Theorem 92 (1) in place of Theorem 68.

There is also a (*b*) form of this theorem which may be proved in an analogous manner.

Theorem 94

If A, B and C be three elements in an optical plane which do not all lie in one general line and if D be the mean of A and B, then a general line through D parallel to BC intersects AC in an element which is the mean of A and C.

Let a_1 be any inertia line through A while b_1 and c_1 are parallel inertia lines through B and C respectively.

Then b_1 and c_1 lie in one inertia plane, say P, c_1 and a_1 in a second inertia plane, say Q, and a_1 and b_1 in a third inertia plane, say R.

Let one of the optical lines through A in the inertia plane Q intersect c_1 in C' and let one of the optical lines through A in the inertia plane R intersect b_1 in B'.

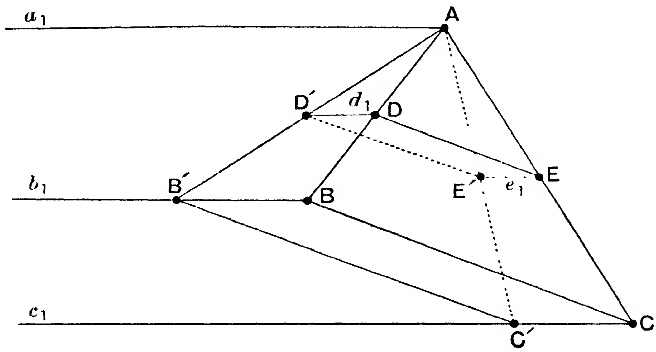


Fig. 30.

Then AC' and AB' may be taken as generators of opposite sets of an inertia plane, say S, containing A, B' and C'.

Let d_1 be an inertia line through D parallel to a_1, b_1 and c_1.

Then d_1 will lie in R and will intersect the optical line AB' in some element, say D'.

If now a general line be taken through D parallel to BC, it will lie in the optical plane, and since the general line AC is distinct from the general line BC it follows from Theorem 87 that this general line through D parallel to BC must intersect AC in some element, say E.

Let e_1 be an inertia line through E parallel to a_1, b_1, c_1, d_1.

Then e_1 will lie in the inertia plane Q and will intersect the optical line AC' in some element, say E'.

Now d_1 and e_1 being parallel inertia lines will lie in an inertia plane, say T, which contains the two intersecting general lines DE and d_1 which are respectively parallel to BC and b_1 in P.

Thus by Theorem 52 the inertia planes T and P are parallel.

But the inertia plane S has the general line $D'E'$ in common with T and the general line $B'C'$ in common with P and so $D'E'$ is parallel to $B'C'$.

Now since A, B and B' lie in the inertia plane R and since D is the mean of A and B and since DD' is parallel to BB', it follows by Theorem 78 provided that A, B and B' do not lie in one general line, that D' is the mean of A and B'.

The only case in which A, B and B' do lie in one general line is when B' coincides with B and then D' is identical with D so that D' is still the mean of A and B'.

Again, since A, B' and C' lie in one inertia plane S and do not all lie in one general line and since D' is the mean of A and B', and $D'E'$ is parallel to $B'C'$, it follows by Theorem 78 that E' is the mean of A and C'.

Further, since A, C and C' lie in one inertia plane Q, since E' is the mean of A and C' and since $E'E$ is parallel to $C'C$, it follows, provided that A, C and C' do not lie in one general line, that E is the mean of A and C.

The only case in which A, C and C' do lie in one general line is when C' coincides with C and then E' coincides with E so that E is still the mean of A and C.

(It is to be noted that we cannot have both B' coinciding with B and C' with C, for then we should have two optical lines AB' and AC' passing through the same element A and lying in an optical plane, which is impossible.) Thus the theorem is proved.

Since there is only one general line through D parallel to BC and this must pass through the mean of A and C, it follows directly that if E be the mean of A and C, then the general line DE is parallel to BC.

Definition. If a pair of parallel general lines in an optical plane be intersected by another pair of parallel general lines, then the four general lines will be said to form a *general parallelogram in the optical plane*.

The terms *corner*, *side line*, *diagonal line*, *adjacent* and *opposite* will be used in a similar sense for the case of a general parallelogram in an optical plane as for one in an inertia plane.

Theorem 95

If we have a general parallelogram in an optical plane, then:

(1) *The two diagonal lines intersect in an element which is the mean of either pair of opposite corners.*

(2) *A general line through the element of intersection of the diagonal lines and parallel to either pair of opposite side lines intersects either of the other side lines in an element which is the mean of the pair of corners through which that side line passes.*

The proof of this theorem is exactly analogous to that of Theorem 79, using Theorem 94 in place of Theorem 78.

Theorem 96

If A, B, C, D be the corners of a general parallelogram in an optical plane, AB and DC being one pair of parallel side lines and BC and AD the other pair of parallel side lines, then if E be the mean of A and B while F is the mean of D and C, the general lines AF and EC are parallel to one another.

The proof of this theorem is exactly analogous to that of Theorem 80, using Theorem 95 in place of Theorem 79 and Theorem 94 in place of Theorem 78.

Theorem 97

If three parallel general lines a, b and c in one optical plane intersect a general line d_1 in A_1, B_1 and C_1 respectively, and intersect a second general line d_2 in A_2, B_2 and C_2 respectively, and if B_1 be the mean of A_1 and C_1, then B_2 will be the mean of A_2 and C_2.

The proof of this theorem is exactly analogous to that of Theorem 81, using Theorem 94 in place of Theorem 78, and Theorem 95 in place of Theorem 79.

Remarks

If P and P' be parallel inertia planes and if a be any generator of P, there is one single generator of P' which is a neutral-parallel of a.

This is easily seen, for if b_1 be any generator of P' belonging to the set not parallel to a and if B be any element in b_1, then either:

(1) B is *before* an element of a,

or (2) B is *after* an element of a,

or (3) B is neither *before* nor *after* any element of a.

In cases (1) and (2), since B does not lie in a and, since b_1 neither intersects a nor is parallel to it, it follows by Post. XII (a and b) that

there is one single element of b_1 which is neither *before* nor *after* any element of a.

Thus there is always an element of b_1 which is neither *before* nor *after* any element of a.

Let B_0 be such an element and let a' be the generator of P' parallel to a and passing through B_0.

Then a' is a neutral-parallel of a.

Again, there can be no other generator of P' besides a' which is a neutral-parallel of a, for any other generator of P' parallel to a' must be either a before- or after-parallel of a' and therefore by Theorem 26 (a or b) such a generator must be a before- or after-parallel of a.

Again, if P and P' be parallel inertia planes and if A be any element of P, while a and b are the two generators of P which pass through A, then there is one single generator of P', say a', which is neutrally parallel to a and there is one single generator of P', say b', which is neutrally parallel to b.

The optical lines a' and b' being generators of opposite sets must intersect in some element, say A'.

Then A' is neither *before* nor *after* any element of a and also neither *before* nor *after* any element of b.

Similarly A is neither *before* nor *after* any element of a' and also neither *before* nor *after* any element of b'.

The elements A and A' will be spoken of as *representatives* of one another in the parallel inertia planes P and P'.

Thus we have the following definition.

Definition. If P and P' be parallel inertia planes and if A and A' be elements in P and P' respectively such that the two generators of P' passing through A' are respectively neutral-parallels of the generators of P which pass through A, then the elements A and A' will be called *representatives* of one another in the parallel inertia planes P and P'.

It is evident that the elements A and A' must lie in a separation line.

THEOREM 98

If P_1 and P_2 be two parallel inertia planes and if A_1 be any element in P_1 while A_2 is its representative in P_2, then if A_1' be any other element in P_1 and A_2' its representative in P_2 the separation lines A_1A_2 and $A_1'A_2'$ are parallel to one another.

Let a_1 and b_1 be the generators of P_1 which pass through A_1 and let a_2 and b_2 be the generators of P_2 which pass through A_2, the optical lines

a_1 and a_2 being neutrally parallel to one another and the optical lines b_1 and b_2 being also neutrally parallel to one another.

Consider first the case where A_1' lies in one of the generators a_1 or b_1 which pass through A_1.

It will be sufficient if we consider the case where A_1' lies in a_1.

Then A_2' will lie in a_2.

Let b_1' be the second generator of P_1 which passes through A_1' and let b_2' be the second generator of P_2 which passes through A_2'.

Then b_1' will be parallel to b_1 while b_2' will be parallel to b_2 and the optical lines b_1' and b_2' will be neutrally parallel to one another by the definition of representative elements.

Now since a_1 and a_2 are neutral-parallel optical lines they determine an optical plane which contains the separation lines A_1A_2 and $A_1'A_2'$ which must therefore either intersect or be parallel to one another.

Now, by Theorem 45, no element of the general line A_1A_2 with the exception of A_1 is either *before* or *after* any element of b_1, and similarly, no element of the general line $A_1'A_2'$ with the exception of A_1' is either *before* or *after* any element of b_1'.

Now suppose, if possible, that A_1A_2 and $A_1'A_2'$ intersect in some element A_0.

Then A_0 could not coincide with either A_1 or A_1' and so would require to be neither *before* nor *after* any element of b_1 and also neither *before* nor *after* any element of b_1'.

If then b_0 were an optical line through A_0 parallel to b_1 and b_1', we should have b_0 neutrally parallel to both b_1 and b_1'.

Thus by Theorem 28, b_1 would require to be neutrally parallel to b_1'.

But b_1 and b_1' are parallel generators of the inertia plane P_1 and so one must be an after-parallel of the other.

Thus the supposition that A_1A_2 and $A_1'A_2'$ intersect leads to a contradiction and therefore is not true.

It follows that A_1A_2 and $A_1'A_2'$ are parallel, which proves the theorem in this special case.

Next consider the case where A_1' does not lie either in a_1 or b_1.

Let b_1' be the generator of P_1 through A_1' parallel to b_1 and let b_2' be the generator of P_2 through A_2' parallel to b_2.

Let b_1' and a_1 intersect in B_1 and let b_2' and a_2 intersect in B_2.

Then since a_1 and a_2 are neutrally parallel and also b_1' and b_2' are neutrally parallel, it follows by the case already proved that A_1A_2 and B_1B_2 are parallel to one another.

Similarly $A_1'A_2'$ and B_1B_2 are parallel to one another.

Thus by Theorem 50 $A_1'A_2'$ and A_1A_2 are parallel to one another.

Sets of three elements which determine optical planes

If A_1, A_2 and A_3 be three distinct elements which do not all lie in one general line and do not all lie in one inertia plane, they either may or may not all lie in one optical plane.

In those cases in which they do lie in an optical plane they determine the optical plane containing them.

We have the following criteria by which we may say that the three elements do lie in one optical plane.

Case I. Three elements A_1, A_2, A_3 lie in one optical plane if A_1 and A_2 lie in an optical line while A_3 is an element which is neither *before* nor *after* any element of the optical line.

This is clearly true since if a be the optical line containing A_1 and A_2, there is an optical line, say b, through A_3 and neutrally parallel to a.

These two optical lines may be taken as generators of an optical plane which will contain A_1, A_2 and A_3.

Now if P be this optical plane it is the only one which contains A_1, A_2 and A_3, for suppose that A_1, A_2 and A_3 also lie in an optical plane P' determined by the two generators a' and b'.

Then, since P' contains A_2 and A_3, it follows by Theorem 86 that P' contains every element of the general line A_2A_3 and since A_2A_3 is a separation line it cannot be parallel to either a' or b' and must therefore intersect both a' and b'.

Again since P' contains A_1 and A_2 it follows that P' contains every element of A_1A_2: that is, it contains the optical line a.

Also since P' contains A_3 it must contain the optical line through A_3 parallel to a: that is, it contains b.

Further a cannot intersect either a' or b' and so must be either parallel to both or identical with one.

Similarly b cannot intersect either a' or b' and so must be either parallel to both or identical with one.

Now every element in the optical plane P must either lie in the separation line A_2A_3 or in a separation line parallel to A_2A_3 and intersecting a and b.

But such a separation line must also intersect a' and b' and will therefore lie in the optical plane P'.

Similarly every element in the optical plane P' must either lie in the

separation line A_2A_3 or in a separation line parallel to A_2A_3 and intersecting a' and b'.

But such a separation line must also intersect a and b and will therefore lie in the optical plane P.

Thus every element in P lies also in P' and every element in P' lies also in P.

Thus the optical planes P and P' are identical and so there is only one optical plane containing the three elements A_1, A_2, A_3.

CASE II. Three elements A_1, A_2, A_3 lie in one optical plane if A_1 and A_2 lie in a separation line while A_3 is an element which is *before one single element* of A_1A_2, or is *after one single element* of A_1A_2.

This may be shown as follows:

Let A_3 be *before* the one single element A_4 of the separation line A_1A_2 and let A_1A_2 be denoted by a.

Then A_3A_4 cannot be an inertia line, for, if it were, we know that it would lie in an inertia plane containing a.

Thus the three elements A_1, A_2, A_3 would lie in one inertia plane, contrary to what was proved on pp. 72–73.

Thus A_3A_4 cannot be an inertia line and so, since A_3 is *before* A_4, it must be an optical line.

Now A_4 must be distinct from at least one of the two elements A_1, A_2, and without loss of generality we may suppose A_4 distinct from A_1.

Then A_1 is neither *before* nor *after* A_4 since they are both elements of the separation line A_1A_2.

Further, A_1 cannot be *before* any element of the optical line A_3A_4 which is *before* A_4, for then A_1 would be *before* A_4, which is impossible.

Similarly A_1 cannot be *after* any element of the optical line A_3A_4 which is *after* A_4.

Again A_1 cannot be *after* any element of the optical line A_3A_4 which is *before* A_4; for if A_5 were such an element of A_3A_4 we should have A_5 *before* two distinct elements of a and so A_5, A_1 and A_4 would lie in one inertia plane which would also contain A_3, contrary to what has already been shown.

Similarly A_1 cannot be *before* any element of the optical line A_3A_4 which is *after* A_4.

Thus A_1 is neither *before* nor *after* any element of the optical line A_3A_4 and so through A_1 there is one single optical line which is neutrally parallel to A_3A_4.

Thus these two optical lines may be taken as generators of an optical plane and, since the separation line a intersects both these optical lines and contains the elements A_1 and A_2, it follows that the three elements A_1, A_2, A_3 lie in an optical plane.

Further, there is only one optical plane containing A_1, A_2 and A_3; for any optical plane containing A_1 and A_2 must also contain A_4, and since, by Case I, there is only one optical plane containing A_3, A_4 and A_1, it follows that there is only one optical plane containing A_1, A_2 and A_3.

Similarly, if A_3 be *after one single element* of the separation line A_1A_2, there is one single optical plane containing the three elements A_1, A_2 and A_3.

Theorem 99

If two optical parallelograms have a pair of opposite corners in common lying in an inertia line, then their separation diagonal lines are such that no element of the one is either before *or* after *any element of the other.*

Let A and B be the two common opposite corners lying in the inertia line a, and let B be *after* A.

Let C and D be the two remaining corners of the one optical parallelogram which we shall suppose to lie in the inertia plane P, and let E and F be the two remaining corners of the other optical parallelogram which we shall suppose to lie in the inertia plane Q.

Then by Theorem 60 the two optical parallelograms have a common centre, say O, which is *after* A and *before* B.

Then the general lines CD and EF are separation lines and so their common element is neither *before* nor *after* any element of either of them.

Let CD be denoted by c and EF by e.

Now, since C and E are two distinct elements in the α sub-set of A which do not lie in one optical line, it follows by Theorem 13 that C is neither *before* nor *after* E, and similarly C is neither *before* nor *after* F.

Let E_1 be any element in e such that E is linearly between O and E_1 and let the optical line through E_1 parallel to EA intersect a in A_1, while the optical line through E_1 parallel to EB intersects a in B_1.

Then by Theorem 72 A is linearly between A_1 and O, while B is linearly between O and B_1.

Thus A_1 must be *before* A and B_1 must be *after* B.

Again, since A_1 is *before* O, and O and E_1 lie in a separation line, we must have A_1 *before* E_1.

Similarly, since B_1 is *after* O, and O and E_1 lie in a separation line, we must have B_1 *after* E_1.

But A is *after* A_1 and C is *after* A and therefore C is *after* A_1, and since A is the only element common to a and the β sub-set of C, it follows that A_1C is an inertia line.

If then C were *before* E_1 it would follow by Theorem 12 that C should

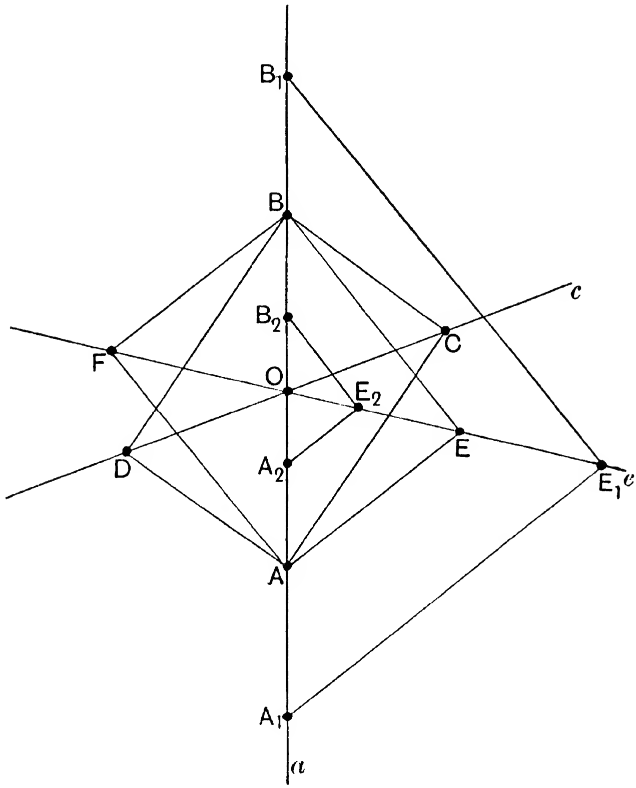


Fig. 31.

lie in the optical line A_1E_1 which it clearly cannot do since A_1C is an inertia line.

Thus C is not *before* E_1.

Further, B is *after* C and B_1 *after* B and therefore B_1 is *after* C and B_1C is an inertia line.

If then C were *after* E_1 it would follow by Theorem 12 that C should lie in the optical line B_1E_1 which it clearly cannot do since B_1C is an inertia line.

Thus C is not *after* E_1.

In a similar manner we may prove that C is neither *before* nor *after* any element F_1 of the separation line e such that F is linearly between O and F_1.

Again let E_2 be any element of e which is linearly between O and E and let the optical line through E_2 parallel to EA intersect a in A_2 while the optical line through E_2 parallel to EB intersects a in B_2.

Then we may prove in a similar manner that E_2 is neither *before* nor *after* C and therefore C is neither *before* nor *after* E_2.

Similarly we may prove that C is neither *before* nor *after* any element F_2 of the separation line e such that F_2 is linearly between O and F.

Thus C is neither *before* nor *after* any element of the separation line e.

Similarly D is neither *before* nor *after* any element of e.

Again if C' be any other element in c distinct from O, then by Theorem 59 there is an optical parallelogram in the inertia plane P having O as centre and C' as one of its corners.

If D' be the corner opposite to C', then D' will also lie in c, and if A' and B' be the remaining two corners these must lie in a.

Then there is one single optical parallelogram in the inertia plane Q having A' and B' as opposite corners.

If E' and F' be the remaining corners of this optical parallelogram, then E' and F' must lie in e.

Thus we have got two new optical parallelograms having a pair of opposite corners A' and B' in common, lying in the inertia line a, while their separation diagonal lines are c and e respectively.

Thus we may prove in a manner similar to that already employed that C' is neither *before* nor *after* any element of e.

Thus no element of c is either *before* or *after* any element of e, as was to be proved.

Remarks

It is evident from the above that any general line which intersects the separation lines c and e in distinct elements must itself be a separation line.

It also appears from this theorem that it is possible to have an element which does not lie in a separation line and which is neither *before* nor *after* any elements of the separation line.

If two distinct elements A_1 and A_2 lie in a separation line a, while A_3 is an element which does not lie in a and is neither *before* nor *after* any element of a, then we have already seen (p. 73) that A_1, A_2 and A_3

cannot lie in an inertia plane and it is also evident that they cannot lie in an optical plane.

For suppose, if possible, that A_3 does lie in an optical plane containing the separation line a; then there would be a generator of the optical plane passing through A_3 and intersecting a in some element, say A_4.

Since A_3 is supposed not to lie in a, the elements A_3 and A_4 would require to be distinct and since they would then lie in an optical line we should have A_3 either *before* or *after* A_4: an element of a, contrary to hypothesis.

Thus A_1, A_2 and A_3 cannot lie in an optical plane.

Again if two distinct elements A_1 and A_2 lie in a separation line while A_3 is an element which does not lie in A_1A_2 and is neither *before* nor *after* any element of A_1A_2, then the element A_2 is neither *before* nor *after* any element of A_3A_1.

For if A_2 were either *before* or *after* any element of A_3A_1, then the three elements A_1, A_2, A_3 would lie either in an inertia or optical plane contrary to what we have just shown.

Similarly A_1 is neither *before* nor *after* any element of A_2A_3.

Again if a be a separation line and A be an element which is not an element of a and is neither *before* nor *after* any element of a, then if B be any element of a, no element of the general line AB is either *before* or *after* any element of a.

This is easily seen, for suppose, if possible, that C is some element of AB which is either *before* or *after* some element of a.

Then C could not lie in a and would lie either in an inertia or optical plane containing a.

But such inertia or optical plane would contain the element A and so the separation line a and the element A would lie in an inertia or optical plane, contrary to what we have already proved.

Thus no element of AB is either *before* or *after* any element of a.

Definition. An inertia line and a separation line which are diagonal lines of the same optical parallelogram will be said to be *conjugate* to one another.

It is evident that if an inertia line and a separation line are conjugate they lie in one inertia plane and intersect one another.

It is also evident that if A be an element lying in an inertia or separation line a in an inertia plane P, then there is only one separation or inertia line through A and lying in P which is conjugate to a; since, if

two optical parallelograms lie in P and have a as a common diagonal line, then their other diagonal lines do not intersect (Post. XVI).

From this it also follows that if two intersecting separation lines b and c be both conjugate to the same inertia line a, then a, b and c cannot lie in the same inertia plane and we shall have a and b in one inertia plane, say P, while a and c lie in another, say Q.

If O be the element of intersection of b and c, then O must lie both in P and Q and therefore in the inertia line a.

If A_1 be any element in a distinct from O, there is one optical parallelogram in the inertia plane P having O as centre and A_1 as one of its corners.

If A_2 be the corner opposite A_1, then there is an optical parallelogram in Q also having A_1 and A_2 as a pair of opposite corners and therefore having the same centre O.

The separation lines b and c will be the separation diagonal lines of the optical parallelograms in P and Q respectively, and so it follows by Theorem 99 that no element of b is either *before* or *after* any element of c.

By considerations similar to the above, we can see that if two intersecting inertia lines b and c be both conjugate to the same separation line a, then a and b must lie in one inertia plane while a and c lie in another distinct inertia plane.

Further if O be the element of intersection of b and c, then O lies in a.

In this case however, since b and c are two intersecting inertia lines, they must lie in one inertia plane which must be distinct from both the others.

Again it is clear that if a be an inertia or separation line lying in an inertia plane P with a separation or inertia line b which is conjugate to a, then any general line c lying in P and parallel to b is also conjugate to a.

Also conversely it is clear that if a be an inertia or separation line lying in an inertia plane P with two distinct separation or inertia lines b and c which are each conjugate to a, then b and c must be parallel to one another.

Theorem 100

If an inertia line a be conjugate to a separation line b, and if an inertia line a' be co-directional with a while a separation line b' is co-directional with b, and if a' and b' intersect one another, then a' is conjugate to b'.

Let P be the inertia plane containing a and b and let O be the element of intersection of a and b, while O' is the element of intersection of a' and b'.

Two cases have to be considered:

(1) O' lies in the inertia plane P.

(2) O' does not lie in the inertia plane P.

Consider first case (1).

Here both a' and b' must lie in P.

Then since a' is co-directional with a and a is conjugate to b and since a, b and a' lie in one inertia plane, it follows that a' is conjugate to b.

Also since b' is co-directional with b and a' is conjugate to b, and since a', b and b' lie in one inertia plane, it follows that a' is conjugate to b'.

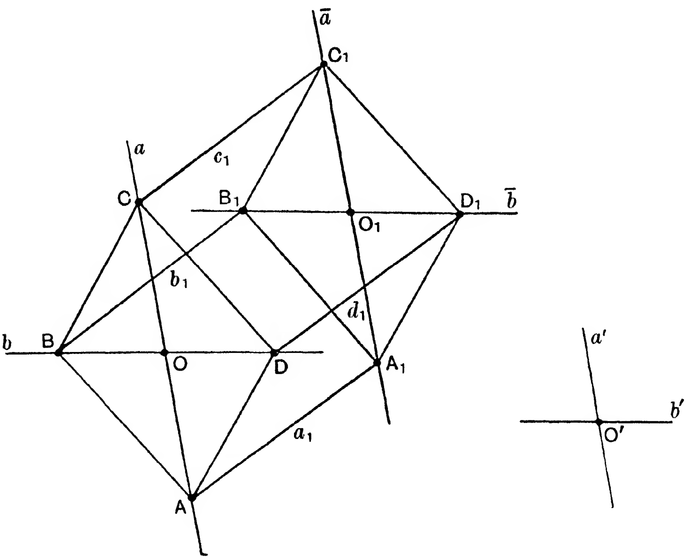


Fig. 32.

Consider next case (2).

Here a' and b' lie in an inertia plane P' which must be distinct from P, since the element O' does not lie in P and therefore, by Theorem 52, P' must be parallel to P.

Now let A be any element of a which is *before* O.

Then we know that there is one single optical parallelogram lying in P which has O as centre and A as one of its corners.

Let C be the corner opposite to A and let B and D be the remaining pair of corners, which must both lie in b, since b is conjugate to a and intersects it in O.

Now, since P and P' are parallel inertia planes and since b is a general

line in P, then, as we showed on p. 76, there is at least one inertia plane, say Q, containing b and another general line, say $\bar{b}$ in P'.

Then $\bar{b}$ must be parallel to b and since b is a separation line, $\bar{b}$ must also be a separation line.

Let b_1 and d_1 be parallel optical lines in Q which pass through B and D respectively and let them intersect the separation line $\bar{b}$ in B_1 and D_1 respectively.

Let an optical line be taken through B_1 parallel to BC and let an optical line be taken through D_1 parallel to DC.

Then these two optical lines will be generators of opposite sets of the inertia plane P' and consequently will intersect in some element, say C_1.

Similarly if an optical line be taken through B_1 parallel to BA and an optical line be taken through D_1 parallel to DA these two optical lines will also lie in P' and will intersect in some element, say A_1.

Now let optical lines a_1 and c_1 be taken through A and C respectively and parallel to b_1 and d_1.

Then C is *after* O and therefore also *after* both B and D and consequently c_1 is an after-parallel of b_1 and also an after-parallel of d_1.

Thus B_1C_1 and D_1C_1 must both intersect c_1 and this latter optical line cannot lie in P' and so cannot have more than one element in common with P'.

But C_1 lies in P' and is the one element common to B_1C_1 and D_1C_1 and so the optical line c_1 must pass through C_1.

In an analogous way we find that a_1 is a before-parallel of both b_1 and d_1 and must pass through the element A_1.

Further, c_1 must be an after-parallel of a_1.

But now, by hypothesis, a is an inertia line so that a and a_1 lie in an inertia plane, which must also contain c_1 since c_1 is parallel to a_1 and passes through the element C of a.

Thus A_1C_1 must be an inertia line parallel to a and we may denote it by $\bar{a}$.

Then $\bar{a}$ and $\bar{b}$ are diagonal lines of the optical parallelogram whose corners are A_1, B_1, C_1 and D_1 and so $\bar{a}$ is conjugate to $\bar{b}$; which intersects it in some element, say O_1.

But a' and $\bar{a}$ are each parallel to a and therefore a' and $\bar{a}$ are co-directional while b' and $\bar{b}$ are each parallel to b and so b' and $\bar{b}$ are co-directional.

Thus, by case (1), a' is conjugate to b', as was to be proved.

Thus the theorem holds in all cases.

Definitions. If A be any element and a be an inertia line not containing A, while B is the element common to a and the α sub-set of A, then we shall speak of B as *the first element of a which is* after A.

Similarly if C be the element common to a and the β sub-set of A, we shall speak of C as *the last element of a which is* before A.

POSTULATE XVIII. **If a, b and c be three parallel inertia lines which do not all lie in one inertia plane* and A_1 be any element in a and if**

B_1 be the first element in b which is after A_1,

C_1 be the first element in c which is after A_1,

B_2 be the first element in b which is after C_1,

C_2 be the first element in c which is after B_1,

then the first element in a which is after B_2 and the first element in a which is after C_2 are identical.

It is evident that there is a (b) form of this postulate in which the word *last* is substituted for the word *first* and the word *before* for the word *after*, but this is not independent, as may be readily seen.

Thus let A_1 be any element in a and let B_1 be the last element in b which is *before* A_1 and let C_2 be the last element in c which is *before* B_1 while A_2 is the last element in a which is *before* C_2.

Then C_2 is the first element in c which is *after* A_2,

B_1 is the first element in b which is *after* C_2,

A_1 is the first element in a which is *after* B_1.

Thus if B_2 be the first element in b which is *after* A_2 and if C_1 be the first element in c which is *after* B_2, it follows by Post. XVIII that the first element in a which is *after* C_1 is identical with the first element in a which is *after* B_1: that is, with the element A_1.

But C_1 is the last element in c which is *before* A_1 and B_2 is the last element in b which is *before* C_1 while A_2 is the last element in a which is *before* B_2.

Thus the last element in a which is *before* B_2 and the last element in a which is *before* C_2 are identical.

* If a, b and c do all lie in one inertia plane, the same result may easily be deduced from the other postulates.

Theorem 101

If an inertia line c be conjugate to two intersecting separation lines d and e, then if A be any element of d and B be any distinct element of e, the general line AB is conjugate to a set of inertia lines which are parallel to c.

Let C_1 be the element of intersection of the separation lines d and e.

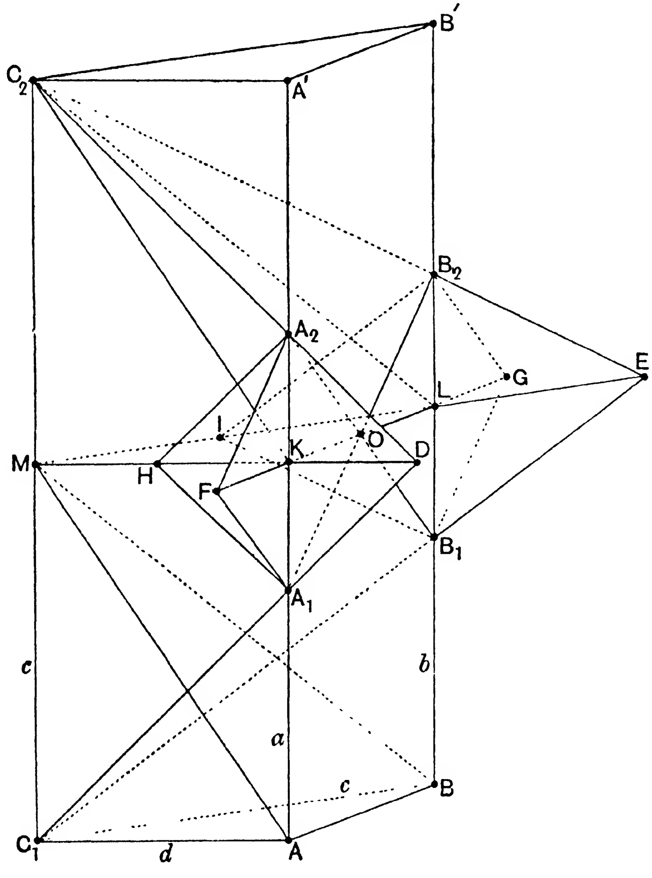


Fig. 33.

Then we know that c and d lie in one inertia plane, say P, while c and e lie in another distinct inertia plane, say Q, and the element C_1 lies in c.

If A or B should coincide with C_1, then the general line AB must coincide with e or d and the result follows directly.

We shall suppose therefore that neither A nor B coincides with C_1.

Then since by Theorem 99 A is neither *before* nor *after* B and since A and B are distinct it follows that AB is a separation line.

Let a be an inertia line through A parallel to c while b is an inertia line through B parallel to c.

Then since A and B lie in a separation line it follows that a and b must be distinct and therefore are parallel to one another.

Thus a and b must lie in an inertia plane which we shall call R.

Further, a must lie in the inertia plane P while b must lie in the inertia plane Q.

Now let A_1 be the first element in a which is *after* C_1,
let B_1 be the first element in b which is *after* C_1,
let A_2 be the first element in a which is *after* B_1,
let B_2 be the first element in b which is *after* A_1.

If now C_2 be the first element in c which is *after* A_2, it follows by Post. XVIII that C_2 is also the first element in c which is *after* B_2.

Now the optical lines C_1A_1 and C_2A_2 cannot be parallel; for since A_1 is *after* C_1 and c and a are parallel inertia lines in the inertia plane P, it would then follow by Theorem 57 (b) that A_2 was *after* C_2.

But we know that C_2 is *after* A_2 and so C_1A_1 and C_2A_2 are not parallel, and, since they lie in one inertia plane, it follows that they must intersect in some element, say D.

Similarly C_1B_1 and C_2B_2 must intersect in some element, say E.

Also since a and b are parallel inertia lines in the inertia plane R and since B_2 is *after* A_1 and A_2 *after* B_1, it follows in a similar manner that the optical lines A_1B_2 and A_2B_1 must intersect in some element, say O.

Now A_2 cannot be identical with A_1 for then we should have the three elements C_1, B_1 and A_1 lying in pairs in three distinct optical lines, which is impossible by Theorem 14.

Further, since B_1 is *after* C_1 and A_2 is *after* B_1, it follows that A_2 is *after* C_1.

But A_2 cannot be *before* A_1 for then we should have A_2 *after* one element of the optical line C_1A_1 and *before* another element of it which would entail that A_2 should lie in the optical line C_1A_1, by Theorem 12.

We know however that A_2 and A_1 are distinct elements in the inertia line a and so A_2 cannot be *before* A_1.

Thus, since A_1 and A_2 are distinct elements in an inertia line and A_2 is not *before* A_1, it follows that A_2 is *after* A_1.

Similarly B_2 is *after* B_1.

Let an optical line through A_1 parallel to DA_2 be taken and an optical line through A_2 parallel to DA_1 and let these intersect in H.

Then A_1, D, A_2, H form the corners of an optical parallelogram in the inertia plane P, having its centre, say K, in the inertia line a.

Again let an optical line through B_1 parallel to EB_2 be taken and an optical line through B_2 parallel to EB_1 and let these intersect in I.

Then B_1, E, B_2, I form the corners of an optical parallelogram in the inertia plane Q, having its centre, say L, in the inertia line b.

If now we take optical parallelograms having C_1 and C_2 as opposite corners in each of the inertia planes P and Q then, by Theorem 60, these have a common centre, say M, lying in the inertia line c.

Also D will be one of the remaining corners of the optical parallelogram in P while E will be one of the remaining corners of the optical parallelogram in Q.

Thus MD and ME will each be conjugate to c.

Further, since the general lines MD and d are both conjugate to c and lie in the same inertia plane P, they must be parallel to one another.

Similarly the general lines ME and e must also be parallel to one another.

But now the optical parallelogram in the inertia plane P having C_1 and C_2 as a pair of opposite corners, and the optical parallelogram whose corners are A_1, D, A_2, H have diagonal lines c and a respectively which do not intersect, and so since they both lie in P their other diagonal lines do not intersect.

But these other diagonal lines have the element D in common and so must be identical.

Thus the general lines MD and KD are identical and so K lies in MD.

Similarly L lies in ME.

Now let an optical line through A_1 parallel to OA_2 be taken and an optical line through A_2 parallel to OA_1 and let these intersect in F.

Then A_1, F, A_2, O form the corners of an optical parallelogram in the inertia plane R, and by Theorem 60 this must have the same centre K as the optical parallelogram whose corners are A_1, D, A_2, H.

Again let an optical line through B_1 parallel to OB_2 be taken and an optical line through B_2 parallel to OB_1 and let these intersect in G.

Then B_1, G, B_2, O form the corners of an optical parallelogram in the inertia plane R, and by Theorem 60 this must have the same centre L as the optical parallelogram whose corners are B_1, E, B_2, I.

But now the optical parallelograms whose corners are A_1, F, A_2, O and B_1, G, B_2, O have the diagonal lines a and b which do not intersect and so, since both lie in the same inertia plane R, their other diagonal lines do not intersect.

That is, FO and GO do not intersect and so since they have the element O in common they must be identical.

Thus O lies in the general line FG; that is, in the general line KL.

Thus KL is conjugate to both a and b.

Now let a general line through C_2 parallel to C_1A intersect a in A', and let a general line through C_2 parallel to C_1B intersect b in B'.

Then A, A', C_2, C_1 form the corners of a general parallelogram in the inertia plane P, while B, B', C_2, C_1 form the corners of a general parallelogram in the inertia plane Q.

Also, since MK and C_2A' are both parallel to C_1A, and since M is the mean of C_1 and C_2, it follows by Theorem 81 that K must be the mean of A and A'.

Similarly L is the mean of B and B'.

Thus by Theorem 80 the general lines AM and KC_2 are parallel to one another and similarly the general lines BM and LC_2 are parallel to one another.

But now, since A_2 is *after* A_1 and since K is the centre of optical parallelograms having A_1 and A_2 as opposite corners, it follows that K is *after* A_1 and *before* A_2.

But, since A_2 is *before* C_2, it follows that K is *before* C_2.

But A_2 is the only element common to a and the β sub-set of C_2 and K is distinct from A_2.

Thus since K is *before* C_2 and does not lie in the β sub-set of C_2, it follows that KC_2 must be an inertia line.

Similarly LC_2 is an inertia line.

Thus KC_2 and LC_2 lie in an inertia plane, say S, while MA and MB being respectively parallel to these must, by Theorem 52, lie in an inertia plane, say S', parallel to S.

But now the general lines KL and AB lie in S and S' respectively and also both lie in the inertia plane R.

Thus AB is parallel to KL and so, since KL is conjugate to a and b, we must also have AB conjugate to a and b, and therefore also conjugate to every inertia line in R parallel to a and b.

But since a and b are parallel to c, therefore all these inertia lines are parallel to c and so the theorem is proved.

*

THEOREM 102

If P_1 and P_2 be parallel inertia planes and if A_1 be any element in P_1 while A_2 is its representative in P_2, then the separation line A_1A_2 is conjugate to every inertia line in P_1 which passes through A_1, and similarly A_1A_2 is conjugate to every inertia line in P_2 which passes through A_2.

Let a_1 and b_1 be the two generators of the inertia plane P_1 which pass through the element A_1, and let a_2 and b_2 be the two generators of the inertia plane P_2 which pass through A_2, and let a_2 be neutrally parallel to a_1 while b_2 is neutrally parallel to b_1.

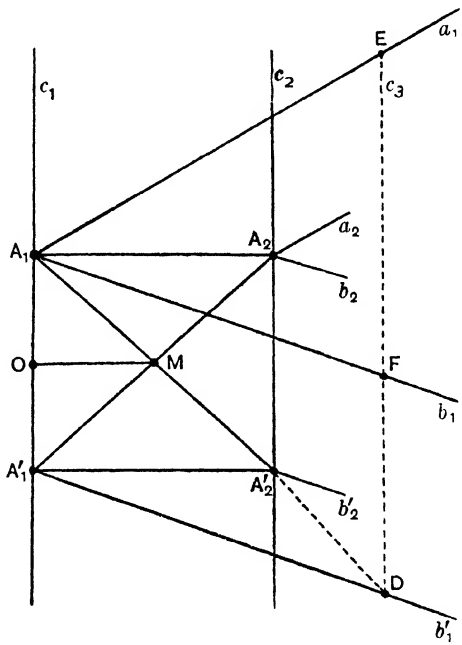


Fig. 34.

Let c_1 be any arbitrary inertia line in P_1 which passes through A_1.

Then c_1 and the element A_2 lie in an inertia plane, say Q, which contains the inertia line c_1 in common with P_1 and the element A_2 in common with P_2 and therefore, by Theorem 46, must have a general line in common with P_2 which must be parallel to c_1 and pass through A_2.

Let this parallel to c_1 through A_2 be denoted by c_2.

Then c_2 must also be an inertia line.

Let A_1' be the one single element common to c_1 and the β sub-set of A_2, while A_2' is the one single element common to c_2 and the β sub-set of A_1, so that A_2A_1' and A_1A_2' are optical lines.

Also A_1' is *before* A_2 while A_2' is *before* A_1.

But, since $A_1 A_2$ is a separation line while $A_1 A_1'$ is an inertia line and A_1' is *before* A_2, it follows that A_1' must also be *before* A_1.

Similarly A_2' must be *before* A_2.

Now A_1' cannot be *after* A_2', for then, since A_1' is before A_1, it would require to lie in the optical line $A_1 A_2'$, which is impossible, since $A_1 A_1'$ is an inertia line.

Similarly A_2' cannot be *after* A_1', and accordingly $A_1'A_2'$ must be a separation line.

Now let b_1' and b_2' be optical lines through A_1' and A_2' respectively parallel to b_1 and accordingly parallel to one another.

We shall presently show that b_1' and b_2' must be neutral parallels, but let us first consider any element D which lies in b_1' and *before* A_1' and let c_3 be an inertia line through D parallel to c_1 and c_2. Let c_3 intersect a_1 in E and b_1 in F.

Then, since D is *before* A_1' and since b_1' and b_1 are parallel optical lines, it follows that F is *before* A_1, so that F lies in the β sub-set of A_1.

Thus E must lie in the α sub-set of A_1 and, since D is *before* A_1', it follows that A_1' is in the α sub-set of D.

Suppose now, if possible, that b_2' is an after-parallel of b_1'.

Then A_2' would be *after* some element of b_1' and so there would be one single element common to b_1' and the β sub-set of A_2'.

This hypothetical element would therefore be *before* A_2' and would also have to be *before* A_1', since $A_1'A_2'$ is a separation line.

If now, we try to identify this hypothetical element with D, we shall find it impossible, for, if we suppose D to be in the β sub-set of A_2' we should have A_2' in the α sub-set of D and accordingly we should have:

A_1' the first element in c_1 which is *after* D;
A_2' the first element in c_2 which is *after* D;
A_1 the first element in c_1 which is *after* A_2';
A_2 the first element in c_2 which is *after* A'_1;
E the first element in c_3 which is *after* A_1;

and so, by Post. XVIII, E should be the first element in c_3 which is *after* A_2.

But A_2 is neither *before* nor *after* any element of a_1, and so E could not be *after* A_2.

Thus the assumption of the existence of an element common to b_1' and the β sub-set of A_2' leads to a contradiction, and so b_2' cannot be an after-parallel of b_1'.

Similarly, if b_2' were supposed to be a before-parallel of b_1', we should require b_1' to be an after-parallel of b_2', and a similar method would show this also to be impossible.

Thus, since b_2' is parallel to b_1', and cannot be either an after- or before-parallel of b_1', it follows that b_2' is a neutral-parallel of b_1'.

But now the separation lines A_1A_2 and $A_1'A_2'$ cannot intersect, for, since A_2 is neither *before* nor *after* any element of b_1, while A_2' is neither *before* nor *after* any element of b_1', it would follow, by Theorem 45, that such an element of intersection, if it existed, would be neither *before* nor *after* any element of either b_1 or b_1', and so there would be an optical line through it which would be neutrally parallel to both b_1 and b_1'.

But, if this were so, it would follow, by Theorem 28, that b_1' was a neutral-parallel of b_1, contrary to what we have already seen, that the element A_1' of b_1' is *before* the element A_1 of b_1.

Thus the separation lines A_1A_2 and $A_1'A_2'$ cannot intersect and so, since they both lie in the inertia plane Q and are distinct, it follows that they are parallel.

Thus A_1, A_2, A_2', A_1' form the corners of a parallelogram in the inertia plane Q and its diagonal lines are A_1A_2' and A_2A_1' which are both optical lines which must intersect in some element, say M.

If then a general line be taken through M parallel to A_1A_2 and meeting A_1A_1' in an element O, it follows, by Theorem 79, that O is the mean of A_1 and A_1'.

Thus an optical parallelogram in the inertia plane Q having A_1 and A_1' as a pair of opposite corners will have OM as its separation diagonal line.

Thus OM is conjugate to c_1 and, since A_1A_2 is parallel to OM and in the same inertia plane Q as are OM and c_1, it follows that A_1A_2 is also conjugate to c_1, and therefore conjugate to every inertia line in P_1 which passes through A_1.

Similarly A_1A_2 is conjugate to every inertia line in P_2 which passes through A_2 and so the theorem is proved.

Theorem 103

If two inertia lines b and c intersect in an element A_1 and are both conjugate to a separation line a, then a is conjugate to every inertia line in the inertia plane containing b and c which passes through the element A_1.

We have already seen that a cannot lie in the inertia plane containing b and c and also that it passes through the element of intersection of b and c.

Let P_1 be the inertia plane containing b and c and let A_2 be any element of a distinct from A_1.

Let P_2 be an inertia plane through A_2 and parallel to P_1.

Let A_2' be the representative of A_1 in the inertia plane P_2.

We shall show that A_2' must be identical with A_2.

Since the inertia line b and the separation line a intersect in the element A_1 they must lie in one inertia plane which contains the inertia line b in common with the inertia plane P_1 and the element A_2 in common with the parallel inertia plane P_2.

Thus the inertia plane containing b and a has a general line, say b', in common with P_2, and b' is parallel to b and is therefore also an inertia line.

Similarly the inertia plane containing c and a has an inertia line, say c', in common with P_2, and c' is parallel to c.

Further b' and c' must both pass through A_2 and must be distinct since b and c are distinct.

Now since A_1 and A_2' are representatives of one another in the parallel inertia planes P_1 and P_2, it follows, by Theorem 102, that the separation line A_1A_2' is conjugate to any inertia line in P_1 which passes through A_1.

Thus A_1A_2' must be conjugate to both b and c.

Suppose now, if possible, that A_2' is distinct from A_2.

Then b is conjugate to both A_1A_2 and A_1A_2' and so, by Theorem 101, b' would be conjugate to A_2A_2'.

Similarly c is conjugate to both A_1A_2 and A_1A_2' and so c' would be conjugate to A_2A_2'.

But then we should have two distinct inertia lines b' and c' both passing through A_2 and conjugate to the same general line A_2A_2' in the inertia plane P_2 which contains b' and c', and this we know is impossible.

Thus A_2' cannot be distinct from A_2 and so A_2 must be the representative of A_1 in the inertia plane P_2.

It follows accordingly that the separation line a is conjugate to every inertia line in P_1 which passes through A_1, and so the theorem is proved.

It is to be noted that in proving the above theorem we have also incidentally proved the following important result:

If two inertia lines b and c intersect in an element A_1 and are both conjugate to a separation line a, then a is such that no element of it,

with the exception of A_1, is either *before* or *after* any element of either of the generators of the inertia plane containing b and c which pass through A_1.

Theorem 104

If an optical line b and an inertia line c intersect in an element A_1 and if a separation line a passing through A_1 be such that no element of a except A_1 is either before *or* after *any element of b and if further a be conjugate to c, then a is conjugate to every inertia line which passes through A_1 and lies in the inertia plane containing b and c.*

Let A_2 be any element of a distinct from A_1 and let b' be an optical line through A_2 parallel to b while c' is an inertia line through A_2 parallel to c.

Then b' must be a neutral-parallel of b.

Let P_1 be the inertia plane containing b and c and let P_2 be the inertia plane containing b' and c'.

Then, since A_2 is neither *before* nor *after* any element of the optical line b, it follows that A_2 does not lie in P_1 and so the inertia planes P_1 and P_2 are parallel to one another.

Let A_2' be the representative of A_1 in P_2; then A_2' must lie in b' by the definition of representative elements and by Theorem 102 A_1A_2' is conjugate to c.

But A_1A_2 is conjugate to c and so if A_2' were distinct from A_2 we should have c conjugate to two intersecting separation lines and so by Theorem 99 no element of A_1A_2 could be either *before* or *after* any element of A_1A_2'.

But if A_2' were distinct from A_2, then, since they each lie in the optical line b', it would follow that the one must be *after* the other.

Thus the supposition that A_2' is distinct from A_2 leads to a contradiction and so A_2' must be identical with A_2.

Thus it follows by Theorem 102 that A_1A_2 (that is a) is conjugate to every inertia line which passes through A_1 and lies in P_1.

Thus the theorem is proved.

It follows directly from the above that no element of a with the exception of A_1 is either *before* or *after* any element of the second generator of P which passes through A_1.

Theorem 105

If a separation line a be conjugate to two intersecting inertia lines b and c, then any inertia line in the inertia plane containing b and c is conjugate to a set of separation lines which are parallel to a.

Let the inertia lines b and c intersect in the element A_1.

Then we know that a must also pass through A_1, but does not lie in the inertia plane containing b and c.

Let P_1 be the inertia plane containing b and c; let A_2 be any element in a distinct from A_1 and let P_2 be an inertia plane through A_2 parallel to P_1.

Then we have seen in the course of proving Theorem 103 that A_1 and A_2 are representatives of one another in the parallel inertia planes P_1 and P_2 respectively, and further every inertia line in P_1 which passes through A_1 is conjugate to a.

Let d be any inertia line in the inertia plane P_1 and let A_1' be any element in d while A_2' is the representative of A_1' in P_2.

Then by Theorem 102 the separation line $A_1'A_2'$ is conjugate to d.

But, provided A_1' be distinct from A_1, it follows by Theorem 98 that $A_1'A_2'$ is parallel to A_1A_2: that is to a, and, since there are an infinite number of elements in d, it follows that d is conjugate to a set of separation lines which are parallel to a.

Thus the theorem is proved.

Theorem 106

If b and c be any two intersecting inertia lines, there is at least one separation line which is conjugate to both b and c.

Let the inertia lines b and c intersect in the element A_1 and let P_1 be the inertia plane containing b and c.

Let any element be taken which does not lie in P_1 and through it let an inertia plane P_2 be taken parallel to P_1.

Let A_2 be the element in P_2 which is the representative of A_1.

Then by Theorem 102 the separation line A_1A_2 is conjugate to any inertia line in P_1 which passes through A_1.

Thus the separation line A_1A_2 is conjugate to both b and c, and so the theorem is proved.

Theorem 107

If b and c be any two intersecting separation lines such that no element of the one is either before *or* after *any element of the other, there is at least one inertia line which is conjugate to both b and c.*

Let the separation lines b and c intersect in the element A_1 and let Q be any inertia plane containing b.

Now, since no element of c is either *before* or *after* any element of b, it follows that c and b do not lie in one inertia plane and therefore c does not lie in Q.

Let d_1 be the inertia line through A_1 in the inertia plane Q which is conjugate to b.

Then d_1 being an inertia line which intersects c, it follows that d_1 and c lie in an inertia plane, say R, which must be distinct from Q.

Let e be any other inertia line in R distinct from d_1 and passing through the element A_1.

Then e being an inertia line which intersects b, it follows that e and b lie in an inertia plane, say Q', which must also be distinct from R.

Let d_1' be the inertia line through A_1 in the inertia plane Q' which is conjugate to b.

Then d_1' may either coincide with e or be distinct from it.

Consider first the case where d_1' coincides with e.

Since then both d_1 and d_1' will lie in the inertia plane R and since the separation line b is conjugate to both d_1 and d_1', it follows by Theorem 103 that b is conjugate to every inertia line in the inertia plane R which passes through the element A_1.

Let a be the inertia line through A_1 in the inertia plane R which is conjugate to c.

Then a must also be conjugate to b and so the theorem will hold in this case.

Consider next the case where d_1' is distinct from e.

Since d_1 and d_1' are intersecting inertia lines, they will lie in an inertia plane, say P_1, which will be distinct from both Q and Q'.

Also, since d_1' does not lie in R in this case, it follows that P_1 is distinct from R, which has the inertia line d_1 in common with P_1.

Since A_1 is the only element of c which is also an element of d_1, it follows that A_1 is the only element of c which lies in P_1.

Let A_0 be any element of c distinct from A_1 and let d_0 and d_0' be inertia lines through A_0 parallel to d_1 and d_1' respectively.

Then, by Theorem 52, d_0 and d_0' lie in an inertia plane, say P_2, which is parallel to P_1.

Again, since A_0 is an element of the inertia plane R and since d_0 is parallel to d_1, it follows that d_0 lies in R.

Further, since the inertia line e is distinct from d_1, it cannot be parallel to d_0 and must therefore intersect d_0 in some element, say B.

But the inertia line e also lies in Q' and so Q' contains the element B in common with P_2.

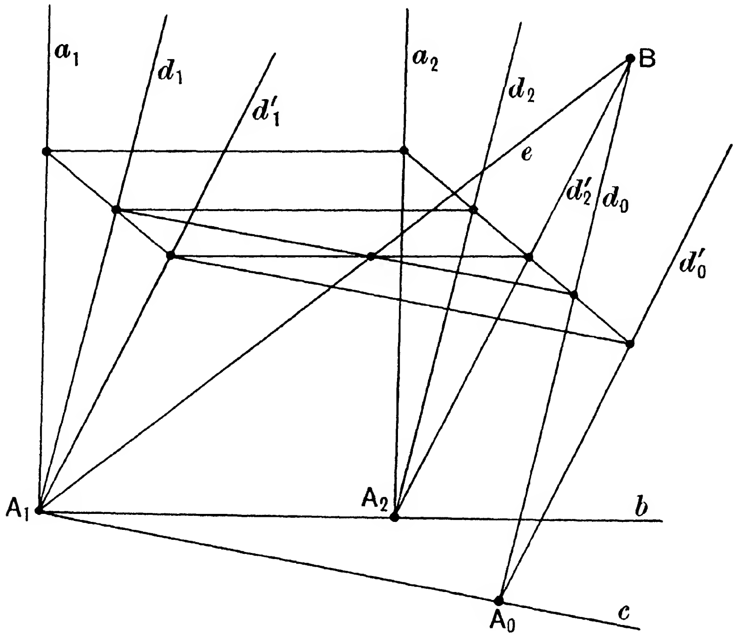


Fig. 35.

Since however Q' has the inertia line d_1' in common with P_1, it follows by Theorem 46 that Q' and P_2 have a general line, say d_2', in common which must be parallel to d_1' and is therefore an inertia line.

Now, since b is a separation line and d_2' an inertia line in the inertia plane Q', it follows that b and d_2' intersect in some element, say A_2.

But A_2 being an element of b is an element of the inertia plane Q, and accordingly Q has the element A_2 in common with the inertia plane P_2.

Since however Q has the inertia line d_1 in common with P_1, it follows by Theorem 46 that Q and P_2 have a general line, say d_2, in common which must be parallel to d_1 and is therefore an inertia line.

But now since d_1 and d_2 are parallel inertia lines in the inertia plane Q and since d_1 is conjugate to b, it follows that d_2 is conjugate to b.

Also since d_1' and d_2' are parallel inertia lines in the inertia plane Q' and since d_1' is conjugate to b, it follows that d_2' is conjugate to b.

Thus the separation line b is conjugate to the two intersecting inertia lines d_2 and d_2' in the inertia plane P_2 and so, by Theorem 103, b is conjugate to every inertia line in P_2 which passes through the element A_2.

Now since no element of c is either *before* or *after* any element of b it follows that the element A_0 is neither *before* nor *after* the element A_2 and therefore, since A_2 and A_0 are distinct, A_2A_0 is a separation line.

Now let a_2 be the inertia line in the inertia plane P_2 which passes through A_2 and is conjugate to A_2A_0.

Then a_2 is also conjugate to b.

Thus if a_1 be an inertia line through A_1 parallel to a_2 it follows by Theorem 101 that a_1 is conjugate to A_1A_0: that is to c.

But a_1 and a_2 being parallel inertia lines through elements of the separation line b and a_2 being conjugate to b, it follows that a_1 is also conjugate to b.

Thus a_1 is conjugate to both b and c and so the theorem is proved.

Theorem 108

If a be a separation line and B be any element which is not an element of a and is neither before *nor* after *any element of a while c is a general line passing through B and parallel to a, then if A be any element of a, while C is an element of c distinct from B, a general line through C parallel to BA will intersect a.*

Let the general line BA be denoted by b and let the general line through C parallel to b be denoted by d.

Then, as was pointed out in the remarks at the end of Theorem 99, no element of b is either *before* or *after* any element of a and so, since a and b intersect in A, it follows by Theorem 107 that there is at least one inertia line which is conjugate to both a and b, and must therefore pass through A.

Let a_1 be such an inertia line and let b_1 be an inertia line through B parallel to a_1, while c_1 is an inertia line through C parallel to a_1 and b_1.

Then a and a_1 lie in an inertia plane which we may call P_a, while a_1, b and b_1 lie in an inertia plane which we may call P_b, and b_1, c and c_1 lie in an inertia plane which we may call P_c.

Then, since B and a do not lie in one inertia plane, it follows that B

is not an element of P_a and so, since b_1 and c are respectively parallel to a_1 and a, it follows that P_c is parallel to P_a.

Now, since b_1 and a_1 both lie in P_o and are parallel to one another, and since a_1 is conjugate to b, it follows that b_1 is also conjugate to b.

But since c is parallel to a and b_1 is parallel to a_1, while c and b_1 intersect in B, it follows, by Theorem 100, that since a_1 is conjugate to a, therefore b_1 is conjugate to c.

Thus b_1 is conjugate to both b and c, which are two distinct and intersecting separation lines and therefore cannot lie in one inertia plane.

Thus C is not an element of P_b and so, if P_d be an inertia plane containing c_1 and d, then, since c_1 is parallel to b_1 while d is parallel to b, it follows that the inertia plane P_d is parallel to P_b.

Then, by Theorem 54, P_a and P_d have a general line in common and, if we call this general line d_1, then, since P_d and P_b are parallel, d_1 must be parallel to a_1 and must be an inertia line.

Now since c_1 is parallel to b_1 and d is parallel to b, and c_1 and d intersect, it follows, by Theorem 100, that, since b_1 is conjugate to b, therefore c_1 is conjugate to d.

Again since b_1 is conjugate to both b and c and since A is an element in b while C is a distinct element in c, it follows, by Theorem 101, that the general line c_1 is conjugate to CA.

Thus c_1 is conjugate to both d and CA.

Now since d is a separation line while d_1 is an inertia line and both lie in the inertia plane P_d, it follows that d and d_1 must intersect in some element, say D.

Thus, since A is an element in CA while D is a distinct element in d, it follows, by Theorem 101, that a_1 is conjugate to DA.

But a_1 is conjugate to the separation line a which also passes through A, and so, since both DA and a lie in the inertia plane P_a which contains a_1, it follows that the general lines DA and a are identical.

Thus D lies in a and therefore the general line d intersects a.

Thus the theorem is proved.

Remarks

If a be a separation line and B be any element which is not an element of a and is neither *before* nor *after* any element of a, then if b be a separation line through B parallel to a, no element of b is either *before* or *after* any element of a.

This is easily seen: for if C were an element of b which was either

before or *after* an element of a, then the separation line a and the element C would lie either in one inertia plane, or in one optical plane.

Such inertia or optical plane would contain the general line through C parallel to a: that is to say it would contain b.

Thus the separation line a and the element B would lie in one inertia or optical plane, which we already know is impossible.

Thus no element of b is either *before* or *after* any element of a and therefore any general line which intersects both a and b must be a separation line.

Again, if AB and DC be two parallel separation lines such that no element of the one is either *before* or *after* any element of the other and if CB is parallel to DA, then no element of DA is either *before* or *after* any element of CB.

This is easily seen: for we know that no element of CB is either *before* or *after* any element of AB and therefore the element A is neither *after* nor *before* any element of CB.

Thus since DA is parallel to CB it follows that no element of DA is either *before* or *after* any element of CB.

Theorem 109

If A and B be two elements lying respectively in two parallel separation lines a and b which are such that no element of the one is either before *or* after *any element of the other, and if A' be a second and distinct element in a, there is only one general line through A' and intersecting b which does not intersect the general line AB.*

We have seen by Theorem 108 that the general line through A' parallel to AB must intersect b.

Let B' be the element of intersection. Then the general lines AB and $A'B'$, being parallel, cannot intersect.

Let any other general line through A' and intersecting b intersect it in the element C.

Then if C should coincide with B the general lines $A'C$ and AB have the element B in common and therefore intersect.

Suppose next that C does not coincide with B.

Since B is neither *before* nor *after* any element of a and since therefore no element of AB is either *before* or *after* any element of a, it follows, by Theorem 107, that there is at least one inertia line, say a_1, which is conjugate to both AB and a and therefore passes through A.

Let b_1 be an inertia line through B parallel to a_1, and let a_1' and b_1' be inertia lines through A' and B' respectively and also parallel to a_1.

Then a_1 and a_1' lie in one inertia plane, say P_1, which contains also the separation line a; while b_1 and b_1' lie in an inertia plane, say P_2, containing b.

Since the elements B, A and A' cannot lie in one inertia plane and since b_1 and b are respectively parallel to a_1 and a, it follows that P_2 is parallel to P_1.

Again a_1 and b_1 lie in an inertia plane, say Q, containing AB, while a_1' and b_1' lie in an inertia plane, say Q', containing $A'B'$.

Since the elements B, A and A' cannot lie in one inertia plane and since a_1' and $A'B'$ are respectively parallel to a_1 and AB, it follows that Q' is parallel to Q.

Now the inertia line a_1' and the element C lie in an inertia plane, say R, and so R has the element C in common with P_2.

Thus, by Theorem 51, R and P_2 have a general line in common, say c_1, which is parallel to a_1' and b_1'.

But now Q is an inertia plane through B, which is an element of P_2 not lying in b_1', and Q is parallel to Q' and therefore, by Theorem 53, the inertia planes R and Q have a general line in common, say f_1, which is parallel to a_1'.

Now f_1 must be an inertia line and therefore will intersect the separation line AB in some element, say F, which must be distinct from A, since otherwise R would coincide with P_1 and could therefore have no element in common with P_2, contrary to hypothesis.

Now, since a_1 is parallel to b_1' while a is parallel to b and, since a_1 is conjugate to a while b_1' and b intersect, therefore b_1' is conjugate to b.

Similarly, since AB is parallel to $A'B'$ and, since a_1 is conjugate to AB while b_1' and $A'B'$ intersect, therefore b_1' is conjugate to $A'B'$.

But now since a_1 is conjugate to the two intersecting separation lines AB and a, and since F is an element in AB, while A' is a distinct element in a, it follows by Theorem 101 that a_1' must be conjugate to $A'F$.

Again since b_1' is conjugate to the two intersecting separation lines $A'B'$ and b, and since A' is an element in $A'B'$ while C is a distinct element in b, it follows in a similar manner that a_1' must be conjugate to $A'C$.

Thus a_1' is conjugate to $A'F$ and to $A'C$ and since $A'F$ and $A'C$ each lie in the inertia plane R and have an element in common, it follows that they must be identical.

Thus F lies in $A'C$ and also in AB and so $A'C$ intersects AB.

Thus there is only one general line through A' and intersecting b which does not intersect the general line AB.

THEOREM 110

If a and b be two parallel separation lines such that no element of the one is either before *or* after *any element of the other, and if one general line intersects a in A and b in B, while a second general line intersects a in A' and b in B', then a general line through any element of AB and parallel to a or b intersects $A'B'$.*

Let D be any element of AB and let d be a general line through D parallel to a or b.

We have to show that d intersects $A'B'$.

If D should coincide with A or B no proof is required and so we shall suppose it distinct from both.

If $A'B'$ be parallel to AB, then no element of AB is either *before* or *after* any element of $A'B'$ and the result follows directly by Theorem 108.

If $A'B'$ be not parallel to AB, then by Theorem 109 the general lines AB and $A'B'$ must intersect in some element, say C.

Now the general lines AB and $A'B'$ being supposed distinct, C must be distinct from at least one of the elements A and B and, without limitation of generality, we may suppose that C is distinct from B.

We shall then have B' distinct from B and so B' will not be an element of AB.

Thus through B' there is a parallel to AB and by Theorem 108 this parallel must intersect d in some element, say D'.

But now $D'B'$ and DB are parallel separation lines such that no element of the one is either *before* or *after* any element of the other and both are intersected by the general lines $D'D$ and $B'C$.

Further since we have supposed D to be distinct from B therefore D' is distinct from B' and so by Theorem 109 there is only one general line through B' and intersecting DB which does not intersect $D'D$.

But $B'B$ being parallel to $D'D$ must be this one general line, and so, since $B'C$ (that is $A'B'$) is distinct from $B'B$, it follows that $A'B'$ intersects $D'D$.

Thus in all cases a general line through any element of AB and parallel to a or b intersects $A'B'$.

Theorem 111

If a and b be two parallel separation lines such that no element of the one is either before *or* after *any element of the other, and if E be any element in a separation line AB which intersects a in A and b in B, and if $A'B'$ be any other separation line intersecting a in A' and b in B' but not parallel to AB, then E either lies in $A'B'$ or in a separation line parallel to $A'B'$ which intersects both a and b.*

If E does not lie in $A'B'$, then by Theorem 110 a separation line through E parallel to a or b intersects $A'B'$ in an element which is neither *before* nor *after* any element of a or b and so, by Theorem 108, a general line through E parallel to $A'B'$ intersects a and also b.

Thus E must lie in a separation line parallel to $A'B'$ and intersecting both a and b when it does not lie in $A'B'$ itself.

Remarks

If a and b be two parallel separation lines such that no element of the one is either *before* or *after* any element of the other and if c and d be any two non-parallel separation lines intersecting both a and b, then it is evident from Theorem 111 that the aggregate consisting of all the elements in c and in all separation lines intersecting a and b which are parallel to c must be identical with the aggregate consisting of all the elements in d and in all separation lines intersecting a and b which are parallel to d.

This follows since each element in the one set of separation lines must also lie in the other set.

Thus the aggregate which we obtain in this way is independent of the particular set of separation lines intersecting a and b which we may select and so we have the following definition.

Definition. If a and b be two parallel separation lines such that no element of the one is either *before* or *after* any element of the other, then the aggregate of all elements of all mutually parallel separation lines which intersect both a and b will be called a *separation plane*.*

If a separation plane P be determined by the two parallel separation lines a and b, then any element C in P must lie in a separation line, say c, which intersects both a and b.

Any other element D in P must either lie in c or in a separation line, say d, parallel to c and intersecting both a and b.

* The name "separation plane" has been adopted from its analogy to a separation line.

If D lies in c, then D is neither *before* nor *after* C.

If D lies in a separation line d parallel to c, we know that no element of d is either *before* or *after* any element of c and so again D is neither *before* nor *after* C.

Thus we have the general result that: *no element of a separation plane is either* before *or* after *any other element of it.*

THEOREM 112

If two distinct elements of a general line lie in a separation plane, then every element of the general line lies in the separation plane.

Let the separation plane be determined by the two parallel separation lines a and b which are such that no element of the one is either *before* or *after* any element of the other.

If the two given elements lie in a separation line which is known to intersect both a and b no proof is required.

Otherwise let C be any element in any separation line AB which intersects a in A and b in B and let D' be any element in any separation line $A'B'$ parallel to AB and intersecting a in A' and b in B'.

We have to show that every element of the general line CD' lies in the separation plane.

Now no element of AB is either *before* or *after* any element of $A'B'$ and so by Theorem 108 a general line through C parallel to a or b will intersect $A'B'$ in some element, say C'.

If D' should coincide with C', then CD' would be parallel to a or b and, since C cannot be either *before* or *after* any element of a or b, it follows that no element of CD' could be either *before* or *after* any element of a or b.

Thus in this case, by Theorem 108, a general line through any element of CD' distinct from C taken parallel to AB will intersect both a and b.

Thus every element of CD' will in this case lie in the separation plane.

If D' should not coincide with C', then since CD' is distinct from CC' and intersects $A'B'$ it follows by Theorem 109 that CD' must intersect both a and b.

Thus again every element of CD' lies in the separation plane determined by a and b.

THEOREM 113

If e be a general line in a separation plane and if A be any element of the separation plane which does not lie in e, then there is one single general line through A in the separation plane which does not intersect e.

We saw in the course of proving Theorem 112 that if a separation plane be determined by two parallel separation lines a and b such that no element of the one is either *before* or *after* any element of the other, then any general line containing two elements in the separation plane and therefore any general line lying in the separation plane must either be parallel to a or b, or else must intersect both a and b.

Suppose first that e intersects both a and b.

Since A does not lie in e it must lie in a separation line d parallel to e and intersecting both a and b.

Now through A there is a separation line, say c, parallel to a or b and which, by Theorem 108, must intersect e and must lie in the separation plane, while any other general line f through A and lying in the separation plane must intersect both a and b.

Thus, by Theorem 109, f being supposed distinct from d must intersect e.

Suppose next that e is parallel to a or b.

Through A there is a separation line parallel to a or b and therefore parallel to e and which, as we know, lies in the separation plane.

Any other general line through A in the separation plane must intersect both a and b and so, by Theorem 110, it must be intersected by e.

Thus there is in all cases one single general line through A in the separation plane which does not intersect e.

THEOREM 114

If A, B and C be three elements in a separation plane which do not all lie in one general line and if D be an element linearly between A and B, while E is an element linearly between A and C, there exists an element which lies both linearly between B and E and linearly between C and D.

The proof of this theorem is quite analogous to that of Theorem 76, the only difference being that V is here a separation plane instead of an inertia plane and, as such, it cannot contain any inertia line.

Thus the words "which does not lie in V" may be omitted from the first sentence of the proof.

Theorem 115

If A, B and C be three elements in a separation plane which do not all lie in one general line and if D be an element linearly between A and B while F is an element linearly between C and D, there exists an element, say E, which is linearly between A and C and such that F is linearly between B and E.

The proof of this theorem is quite analogous to that of Theorem 77, the only difference being that V is here a separation plane instead of an inertia plane and, as such, it cannot contain any inertia line.

Thus the words "which does not lie in V" may be omitted from the first sentence of the proof.

Remarks

It will be observed that Theorem 114 is the analogue of Peano's axiom (14) for the case of elements in a separation plane, while Theorem 115 is the corresponding analogue of his axiom (13).

Further, Theorem 113 corresponds to the Euclidean axiom of parallels for the case of general lines in a separation plane.

Theorem 116

If A, B and C be three elements in a separation plane which do not all lie in one general line and if D be an element linearly between A and B while DE is a general line through D parallel to BC and intersecting AC in the element E, then E is linearly between A and C.

The proof of this theorem is exactly analogous to that of Theorem 90, using Theorem 114 in place of Theorem 88, and Theorem 115 in place of Theorem 89.

Theorem 117

If three parallel general lines a, b and c in one separation plane intersect a general line d_1 in A_1, B_1 and C_1 respectively and intersect a second general line d_2 in A_2, B_2 and C_2 respectively, then if B_1 is linearly between A_1 and C_1 we shall also have B_2 linearly between A_2 and C_2.

The proof of this theorem is exactly analogous to that of Theorem 91, using Theorem 116 in place of Theorem 90.

THEOREM 118

If A, B and C be three elements in a separation plane which do not all lie in one general line and if D be the mean of A and B, then a general line through D parallel to BC intersects AC in an element which is the mean of A and C.

The proof of this theorem is exactly analogous to that of Theorem 94. It is to be noted however that for the case of a separation plane we can never have B' coinciding with B or C' coinciding with C, since a separation plane cannot contain an optical line.

Since there is only one general line through D parallel to BC and this must pass through the mean of A and C, it follows directly that if E be the mean of A and C, then the general line DE is parallel to BC.

Definition. If a pair of parallel general lines in a separation plane be intersected by another pair of parallel general lines, then the four general lines will be said to form a *general parallelogram in the separation plane.*

The terms *corner*, *side line*, *diagonal line*, *adjacent* and *opposite* will be used in a similar sense for the case of a general parallelogram in a separation plane as for one in an inertia or optical plane.

THEOREM 119

If we have a general parallelogram in a separation plane, then:

(1) *The two diagonal lines intersect in an element which is the mean of either pair of opposite corners.*

(2) *A general line through the element of intersection of the diagonal lines and parallel to either pair of opposite side lines intersects either of the other side lines in an element which is the mean of the pair of corners through which that side line passes.*

The proof of this theorem is exactly analogous to that of Theorem 79, using Theorem 118 in place of Theorem 78.

THEOREM 120

If A, B, C, D be the corners of a general parallelogram in a separation plane; AB and DC being one pair of parallel side lines and BC and AD the other pair of parallel side lines, then if E be the mean of A and B, while F is the mean of D and C, the general lines AF and EC are parallel to one another.

The proof of this theorem is exactly analogous to that of Theorem 80, using Theorem 119 in place of Theorem 79, and Theorem 118 in place of Theorem 78.

Theorem 121

If three parallel general lines a, b and c in one separation plane intersect a general line d_1 in A_1, B_1 and C_1 respectively and intersect a second general line d_2 in A_2, B_2 and C_2 respectively, and if B_1 be the mean of A_1 and C_1, then B_2 will be the mean of A_2 and C_2.

The proof of this theorem is exactly analogous to that of Theorem 81, using Theorem 118 in place of Theorem 78, and Theorem 119 in place of Theorem 79.

Sets of three elements which determine separation planes

If A_1, A_2 and A_3 be three distinct elements which do not all lie in one general line and do not all lie in one inertia plane or in one optical plane, then they must all lie in one separation plane, as we shall shortly show.

In those cases in which they do all lie in one separation plane they determine the separation plane containing them.

We have the following criterion by which we may say that the three elements do lie in one separation plane.

Three elements A_1, A_2, A_3 lie in one separation plane if A_1 and A_2 lie in a separation line while A_3 is an element which is not an element of the separation line and is neither *before* nor *after* any element of the separation line.

This is clearly true since if a be the separation line containing A_1 and A_2, there is a separation line b through A_3 and parallel to a which is such that no element of b is either *before* or *after* any element of a.

The separation lines a and b then determine a separation plane which will contain A_1, A_2 and A_3.

If P be this separation plane it is the only one which contains A_1, A_2 and A_3, for suppose A_1, A_2 and A_3 also lie in a separation plane P' determined by the two parallel separation lines a' and b', which are such that no element of b' is either *before* or *after* any element of a'.

Now since P' contains A_1, A_2 and A_3 it must contain the three general lines A_1A_2, A_2A_3 and A_3A_1, by Theorem 112.

At most only one of these general lines could be parallel to a' or b'.

Suppose first that A_1A_2 or a is not parallel to a' or b'.

Then a must intersect both a' and b', and since A_3 is an element of P' the separation line b through A_3 parallel to a must lie in P' and must intersect both a' and b'.

Then every element in P must lie in a separation line intersecting both a and b and parallel to a' or b'.

But we know that every element of any such separation line c must lie in P', for by Theorem 110 a general line through any element of c parallel to a or b must intersect a' and b'.

Similarly every element in P' must lie in P and so P' must be identical with P.

Next suppose that a is parallel to a' or b'.

Then A_1A_3 cannot be parallel to a' or b' and so must intersect both a' and b'.

Then any element in P must lie either in A_3A_1 or in a general line parallel to A_3A_1 and intersecting both a and b.

But any such general line must also intersect both a' and b', and so every element in P must also lie in P', and similarly every element in P' must also lie in P.

Thus again P' must be identical with P.

Thus there is only one separation plane containing the three elements A_1, A_2 and A_3.

Any three distinct elements A_1, A_2 and A_3 which do not all lie in one general line must all lie either in an inertia plane, an optical plane, or a separation plane.

This is easily seen; for A_1 and A_2 must lie either in an optical line, an inertia line, or a separation line.

If A_1A_2 be an optical line we must have either

(1) A_3 *after* an element of A_1A_2,

or (2) A_3 *before* an element of A_1A_2,

or (3) A_3 neither *before* nor *after* any element of A_1A_2.

We cannot have A_3 *after* one element of A_1A_2 and *before* another element of it, since A_3 is not an element of A_1A_2 (Theorem 12).

In cases (1) and (2), as we have seen, A_1, A_2 and A_3 lie in an inertia plane.

In case (3) we have seen that A_1, A_2 and A_3 lie in an optical plane.

If A_1A_2 be an inertia line we know that the three elements must always lie in an inertia plane.

If A_1A_2 be a separation line we must have either

(1) A_3 *after* at least two distinct elements of A_1A_2,

or (2) A_3 *before* at least two distinct elements of A_1A_2,

or (3) A_3 *after* one single element of A_1A_2,

or (4) A_3 *before* one single element of A_1A_2,

or (5) A_3 neither *before* nor *after* any element of A_1A_2.

We cannot have A_3 *after* one element of A_1A_2 and *before* another

element of it for then we should have one element of $A_1 A_2$ *after* another element of it, contrary to the hypothesis that $A_1 A_2$ is a separation line.

We have already seen that in cases (1) and (2) A_1, A_2 and A_3 lie in an inertia plane.

Also in cases (3) and (4) we have seen that A_1, A_2 and A_3 lie in an optical plane.

Finally in case (5) we have seen that A_1, A_2 and A_3 lie in a separation plane.

This exhausts all the possibilities which are logically open and so we see that A_1, A_2 and A_3 must always lie either in an inertia plane, an optical plane, or a separation plane.

It follows directly that any two intersecting general lines a and b must lie either in an inertia plane, an optical plane, or a separation plane, which we may call P.

Any element in P must lie either in b or in a general line parallel to b and intersecting a.

Also, conversely, any element in b or in any general line which intersects a and is parallel to b, must lie in P.

Thus we have the following definition:

Definition. If a and b be any two intersecting general lines, then the aggregate of all elements of the general line b and of all general lines parallel to b which intersect a will be called a *general plane*.

Thus a general plane is a common designation for an inertia plane, an optical plane, or a separation plane.

By combining Theorems 76, 88 and 114 we now see that the analogue of Peano's axiom (14) holds in general for our geometry; while by combining Theorems 77, 89 and 115 we see that the analogue of his axiom (13) also holds in general.

Again by combining Theorems 47, 87 and 113 we get what corresponds to the Euclidean axiom of parallels for the case of general lines in a general plane.

Peano's fifteenth axiom is as follows:

A point can be found external to any plane.

It is evident in our geometry that, since there is more than one general plane, there is an element external to any general plane, and so the analogue of Peano's axiom (15) also holds.

If a and b be two intersecting general lines in a general plane P and if through any element A not lying in P two general lines a' and b' be taken respectively parallel to a and b, then if P' be the general plane

determined by a' and b', the two general planes P and P' can have no element in common.

This is easily seen, for in the first place the general line a' can have no element in common with P, for then, since it is parallel to a, every element of a' would have to lie in P, contrary to the hypothesis that the element A does not lie in P.

Similarly b' can have no element in common with P.

If now B be any element in a' distinct from A and if b'' be a general line through B parallel to b', then b'' must also be parallel to b and, since B does not lie in P, it follows that b'' can have no element in common with P.

But any element in P' must lie either in b' or in a general line parallel to b' which intersects a' and therefore the general plane P' can have no element in common with P.

Theorem 122

If a and b be any two intersecting general lines in a general plane P and if through any element O' not lying in P two general lines a' and b' respectively parallel to a and b be taken determining a general plane P', then there is a general line through O' and lying in P' which is parallel to any general line in P.

Let the general lines a and b intersect in the element O and let A and B be any two elements distinct from O and lying in a and b respectively.

Then the general lines OO' and a determine a general plane which must contain a', since a' is parallel to a and intersects OO'.

Thus a general line through A parallel to OO' will intersect a' in some element, say A'.

Similarly a general line through B parallel to OO' will intersect b' in some element, say B'.

Then BB' will be parallel to AA'.

But AB and AA' determine a general plane which must contain BB' and so the general lines AB and $A'B'$ must lie in one general plane.

But AB lies in P while $A'B'$ lies in P', and so $A'B'$ can have no element in common with AB and must therefore be parallel to it.

Let the general line AB be denoted by c and the general line $A'B'$ by c'.

Let c_1 be a general line through O parallel to c while c_1' is a general line through O' parallel to c'.

Then c_1 will lie in P and c_1' will lie in P', and since c' is parallel to c we must also have c_1' parallel to c_1.

Now any general line in P and passing through O, with the exception of c_1, must intersect c in some element, say X.

If X should coincide with either A or B, we know that $O'A'$ and $O'B'$ are respectively parallel to OA and OB, so that we shall suppose X distinct from A and B.

If now a general line be taken through X parallel to AA', such general line will lie in the general plane determined by AB and AA' and will therefore intersect $A'B'$ in some element, say X'.

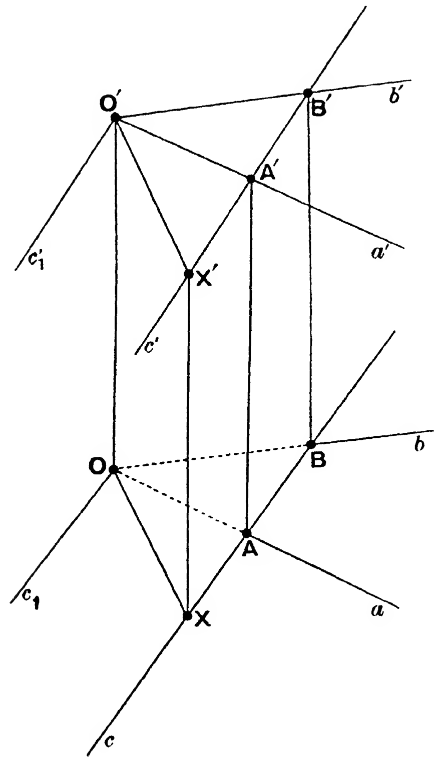


Fig. 36.

Now XX' must be parallel to OO' and so XX' must lie in the general plane determined by OX and OO'.

Thus OX and $O'X'$ lie in one general plane.

But OX lies in P while $O'X'$ must lie in P' and, since P and P' have no element in common, it follows that $O'X'$ is parallel to OX.

Thus through O' there is a general line in P' which is parallel to any general line in P which passes through O, and since any general line in P which does not pass through O is parallel to one which does pass

through O, it follows that there is a general line through O' and lying in P' which is parallel to any general line in P.

It also follows directly from the above that through *any* element of P' there is a general line in P' which is parallel to any general line in P.

REMARKS

We have already given a definition of the parallelism of inertia planes and are now in a position to give a definition of the parallelism of general planes which will include that of inertia planes as a special case.

Definition. If P be a general plane and if through any element A outside P two general lines be taken respectively parallel to two intersecting general lines in P, then the two general lines through A determine a general plane which will be said to be *parallel* to P.

Theorem 52 shows that this definition agrees with that given for the case of inertia planes.

If P be a general plane and A be any element outside it, while P' is a general plane through A parallel to P, then it is evident from Theorem 122 that, since P' contains the general line through A parallel to any general line in P, the general plane P' must be uniquely determined when we know P and A.

Thus *through any element outside a general plane P there is one single general plane parallel to P.*

Also it is clear that this general plane must be of the same kind as P.

Again, since two distinct general lines which are parallel to a third general line are parallel to one another, it follows that: *two distinct general planes which are parallel to a third general plane are parallel to one another.*

Definition. If P be a general plane and if through any element A outside P a general line a be taken parallel to any general line in P, then the general line a will be said to be *parallel* to the general plane P.

THEOREM 123

If a general plane P have one element in common with each of a pair of parallel general planes Q and R, then, if P have a second element in common with Q it also has a second element in common with R.

Let the general plane P have the element A in common with Q and the element A' in common with R.

Further let P and Q have a second element B in common.

Then, as was observed at the end of Theorem 122, there is a general line, say c, through A' in the general plane R which is parallel to AB.

But c must also lie in P, and so any element of c distinct from A' is a second element common to P and R.

Thus the theorem is proved.

Theorem 124

If two parallel general lines a and b lie in one general plane R and if two other distinct general planes P and Q containing a and b respectively have an element A_1 in common, then P and Q have a general line in common which is parallel to a and b.

The proof of this theorem is exactly analogous to that of Theorem 51, using Theorem 123 in place of Theorem 46.

Theorem 125

If three distinct general planes P, Q and R and three parallel general lines a, b and c be such that a lies in P and R, b in Q and P and c in R and Q, then if Q' be a general plane parallel to Q through some element of P which does not lie in b the general planes R and Q' have a general line in common which is parallel to c.

The proof of this theorem is analogous to that of Theorem 53, using Theorem 123 in place of Theorem 46.

Since however a general plane does not always contain an optical line, we take any general line through the element A distinct from a, which lies in the general plane P, and such general line must intersect b in an element which we shall call B.

We then take any general line through B distinct from b, which lies in the general plane Q and this general line must intersect c in some element which we shall call C.

Then BA and BC lie in a general plane which we shall call S.

The demonstration from this point on is similar to that of Theorem 53, once more using Theorem 123 in place of Theorem 46.

If a pair of parallel general lines be both intersected by another pair of parallel general lines then the four general lines will form a *general parallelogram* either in an *acceleration plane*, an *optical plane*, or a *separation plane*.

Thus a *general parallelogram* may now be defined in this way without specifying which type of general plane it lies in.

THEOREM 126

If two general parallelograms have a pair of adjacent corners A and B in common, their remaining corners either lie in one general line or else form the corners of another general parallelogram.

Let A, B, C, D be the corners of the one general parallelogram and A, B, C', D' the corners of the other, and let AC and BD be a pair of opposite side lines of the first general parallelogram while AC' and BD' are a pair of opposite side lines of the second.

Then CD and $C'D'$ being each parallel to AB must either be parallel to one another or else must be identical.

In the latter case the corners C, D, C', D' lie in one general line.

Suppose now that CD and $C'D'$ are distinct and therefore parallel; we have to prove that CC' is parallel to DD'.

Two cases have to be considered:

(1) The two general parallelograms lie in distinct general planes,

or (2) The two general parallelograms lie in the same general plane.

We shall first consider case (1).

Since CD and $C'D'$ are parallel they must lie in a general plane, say P.

Again AC and AC' must lie in a general plane, say Q, distinct from the general planes of either of the general parallelograms, since by hypothesis C' does not lie in the general plane containing A, B, C and D.

Similarly BD and BD' must lie in a general plane, say R, distinct from the general planes of either of the general parallelograms.

Further, the element A cannot lie in R, since otherwise A, B, C, D, C' and D' would all lie in one general plane, contrary to hypothesis.

But AC is parallel to BD, while AC' is parallel to BD' and therefore Q is parallel to R.

Thus the general lines CC' and DD' can have no element in common, and since they both lie in P, it follows that they are parallel.

Thus C, C', D', D form the corners of another general parallelogram.

We have next to consider case (2).

Let P be the general plane containing the two given general parallelograms, and let Q be any other general plane distinct from P and containing the general line AB.

Let AC_1 be any general line distinct from AB which passes through A and lies in Q.

Through any element C_1 of AC_1 distinct from A let a general line be taken parallel to AB and let it meet the general line through B parallel to AC_1 in the element D_1.

Then by the case already proved the general lines CC_1 and DD_1 are parallel.

Similarly $C'C_1$ and $D'D_1$ are parallel to one another.

But now the general parallelograms whose corners are C_1, D_1, D, C and C_1, D_1, D', C' cannot lie in one general plane; for the general lines CD and $C'D'$ both lie in P, while C_1D_1 does not lie in P.

Thus again by case (1) CC' is parallel to DD' and so C, C', D', D form the corners of a general parallelogram.

Thus the theorem holds in all cases.

Theorem 127

(1) *If three distinct elements A, B and C in a general plane P do not all lie in one general line and if D be any element linearly between B and C, then any general line passing through D and lying in P and which is distinct from BC and AD must either intersect AC in an element linearly between A and C, or else intersect AB in an element linearly between A and B.*

(2) *If further E be an element linearly between C and A and if F be an element linearly between A and B, then D, E and F cannot lie in one general line.*

In order to prove the first part of the theorem let a be any general line passing through D and lying in P.

Then a must either be parallel to AC or else intersect AC in some element, say E.

If a be parallel to AC, then it follows by Theorems 72, 90 and 116 that a must intersect AB in an element which is linearly between A and B.

If a intersects AC in an element E, then provided a be distinct from BC and AD we must either have:

(i) E linearly between A and C,
or (ii) C linearly between A and E,
or (iii) A linearly between C and E.

In case (ii) it follows by the analogue of Peano's axiom (13) that a intersects AB in an element linearly between A and B, while in case (iii) it follows by the analogue of Peano's axiom (14) that a intersects AB in an element linearly between A and B.

Thus the first part of the theorem is proved.

In order now to prove the second part of the theorem it is to be observed in the first place that since the elements D, E and F lie in

three distinct general lines BC, CA and AB and are distinct from the elements of intersection of these, therefore the elements D, E and F are all distinct.

If then D, E and F lay in one general line, we should require to have either:

E linearly between D and F,
or F linearly between E and D,
or D linearly between F and E.

Now the elements F, C and B do not lie in one general line and we have D linearly between B and C.

If then we had also E linearly between D and F it would follow that A must be linearly between B and F, contrary to the hypothesis that F is linearly between A and B.

Thus E cannot be linearly between D and F.

Similarly F cannot be linearly between E and D, and further D cannot be linearly between F and E.

It follows therefore that D, E and F cannot lie in one general line and so the second part of the theorem is proved.

Theorem 128

If an inertia line a be conjugate to two intersecting separation lines b and c, then b and c lie in a separation plane such that any separation line in it is conjugate to a set of inertia lines which are parallel to a.

Let the separation lines b and c intersect in the element A.

Then we know that a must also pass through A and that the separation lines b and c must be such that no element of the one is either *before* or *after* any element of the other, and so there must be a separation plane, say P, which contains them.

Let B and C be elements in b and c respectively and let them both be distinct from A.

Then BC is a separation line which we may call d and which lies in the separation plane P.

Let e be an inertia line through B parallel to a.

Then e is conjugate to b and, by Theorem 101, it must also be conjugate to d.

Now we know that there is only one general line in P and passing through A which does not intersect d.

Let AF be any general line passing through A and intersecting d in F.

Then, by Theorem 101, since e is conjugate to b and d, it follows that a must be conjugate to AF.

Again, if d' be the general line through A parallel to d, it must lie in the separation plane P, and, since e is conjugate to d, while a and d' are respectively parallel to e and d and, since a and d' intersect one another, it follows by Theorem 100 that a must be conjugate to d'.

Thus every separation line passing through A in the separation plane P is conjugate to a and therefore also conjugate to any inertia line which intersects it and is parallel to a.

Consider now any separation line f in P which does not pass through A.

Then there is a separation line f' passing through A and parallel to f, and a must be conjugate to f'.

Thus by Theorem 100 any inertia line intersecting f and parallel to a must be conjugate to f.

Thus any separation line in P, whether it pass through A or not, must be conjugate to a set of inertia lines which are parallel to a, and so the theorem is proved.

Theorem 129

If O be any element in a separation line b lying in a separation plane P and if a be an inertia line through O which is conjugate to every separation line in P which passes through O, then there is one and only one such separation line which is conjugate to every inertia line passing through O and lying in the inertia plane containing a and b.

Let Q be the inertia plane containing a and b and let Q' be an inertia plane parallel to Q through any element of P which does not lie in b.

Then by Theorem 123 P and Q' will have a general line, say b', in common which must be parallel to b and must be a separation line.

Let c be one of the generators of Q which pass through O.

Then since Q' is parallel to Q there is one single generator of Q', say c', which is neutrally parallel to c.

Let c' intersect b' in O'.

Then O' is neither *before* nor *after* any element of c and so no element of the general line OO' with the exception of O is either *before* or *after* any element of c.

But OO' lies in P and therefore is conjugate to a, and so, by Theorem 104, OO' is conjugate to every inertia line in Q which passes through O.

Thus, as in Theorem 104, O and O' are representatives of one another in the parallel inertia planes Q and Q', and further, we may show as in

Theorem 103 that if O'' be any element of Q' distinct from O' the general line OO'' cannot be conjugate to two distinct inertia lines in Q which pass through O.

But now any separation line in P which passes through O must either be identical with b or else intersect b' in some element.

If it should intersect b' in any element other than O' it cannot be conjugate to more than one inertia line in Q which passes through O.

Also if it be identical with b it cannot be conjugate to more than one inertia line in Q which passes through O.

Thus there is one and only one separation line in P which passes through O and is conjugate to every inertia line passing through O and lying in the inertia plane Q.

THEOREM 130

If a separation line a have an element O in common with an inertia plane P and be conjugate to every inertia line in P which passes through O, and if c be any such inertia line and b be the separation line in P which passes through O and is conjugate to c, then b is conjugate to every inertia line in the inertia plane containing a and c which passes through the element O.

Let A_1 be any element in a distinct from O, and let d be any inertia line in P which passes through O and is distinct from c.

Let B_1 be the one single element common to d and the α sub-set of A_1.

Then A_1B_1 is an optical line and B_1 is *after* A_1 and so, since A_1O is a separation line while B_1O is an inertia line, we must have B_1 *after* O.

Let D be the one single element common to c and the α sub-set of B_1 and let E be the one single element common to c and the β sub-set of B_1.

Then B_1D and B_1E are optical lines lying in P.

Also D is *after* B_1 while B_1 is *after* both A_1 and O and so D is *after* both A_1 and O.

Let the optical line through O parallel to B_1D intersect the optical line through D parallel to B_1E in F and let the optical line through E parallel to B_1D intersect DF in B_2.

Then B_1, E, B_2, D are the corners of an optical parallelogram lying in P. Let C be its centre.

Then, since DF and OF are both optical lines and since D is *after* O but is not in an optical line with it, it follows that F is *after* O.

But now since A_1 is not an element of the optical line DF but is

before an element of it, it follows that there is one single element common to the optical line DF and the α sub-set of A_1.

Let B_2' be this element, which we shall prove must be identical with B_2.

Then, since D is *after* A_1 but is not in an optical line with it, it follows that B_2' cannot be either identical with D or *after* D and therefore, since B_2' and D lie in an optical line, it follows that B_2' is *before* D.

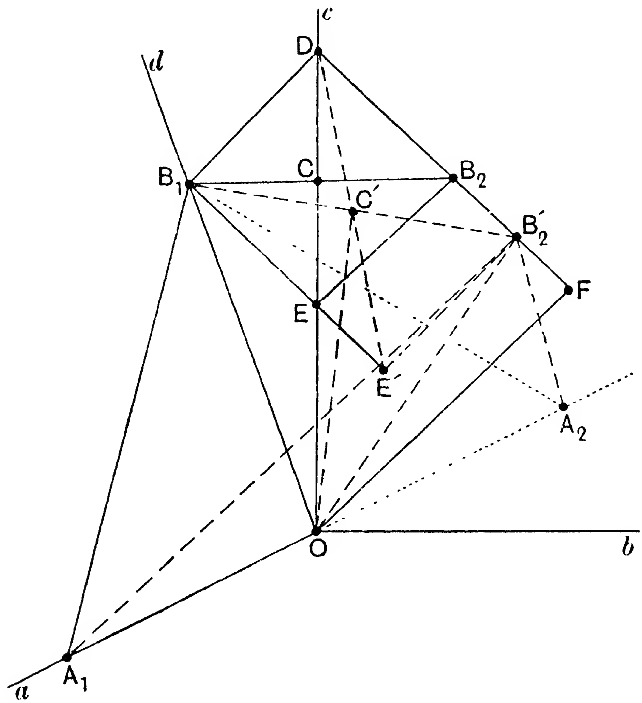


Fig. 37.

But B_1 and D lie in another optical line and B_1 is also *before* D and therefore B_2' is neither *before* nor *after* B_1, so that $B_1 B_2'$ is a separation line.

Again, since OF is one of the generators of P which pass through O and since by hypothesis $A_1 O$ is conjugate to every inertia line in P which passes through O and A_1 is distinct from O, it follows that A_1 is neither *before* nor *after* any element of OF.

Thus A_1 is not *before* F and so B_2' can neither be *before* F nor identical with it and so, since B_2' and F lie in an optical line, it follows that B_2' is *after* F.

But, since F is *after* O, it follows that B_2' is *after* O and, since B_2' and O do not lie in one optical line, it follows that $B_2'O$ must be an inertia line.

Let the optical line through B_2' parallel to DB_1 intersect B_1E in E'. Then B_1, E', B_2', D form the corners of an optical parallelogram of which B_1B_2' is the separation diagonal line, and accordingly, $E'D$ is the inertia diagonal line.

Let C' be the centre of this optical parallelogram.

Then C' is linearly between B_1 and B_2' and so, since O is *before* both B_1 and B_2' and is not in the general line B_1B_2', it follows by Theorem 73 that $C'O$ is an inertia line.

Now in the inertia plane containing a and d take the second optical line which passes through B_1 and let it intersect a in the element A_2.

Then, since OB_1 is conjugate to a, it follows that A_1, B_1 and A_2 are three corners of an optical parallelogram having O as its centre.

But we showed that OB_2' must be an inertia line and, as it lies in P and passes through O, it must also be conjugate to a.

But O is the mean of A_1 and A_2 while A_1B_2' is an optical line and so A_2B_2' must also be an optical line.

But now, since C' is the mean of B_1 and B_2', it follows that B_1, A_1 and B_2' are three corners of an optical parallelogram of which C' is the centre and so B_1B_2' is conjugate to $C'A_1$.

Similarly B_1, A_2 and B_2' are three corners of an optical parallelogram of which C' is the centre and so B_1B_2' is conjugate to $C'A_2$. Further, since B_1B_2' is a separation line, it follows that $C'A_1$ and $C'A_2$ are both inertia lines.

Thus B_1B_2' is conjugate to two inertia lines passing through the element C' and therefore it must be conjugate to every inertia line passing through C' and lying in the inertia plane containing $C'A_1$ and $C'A_2$.

But the element O lies in A_1A_2 and $C'O$ must therefore lie in this inertia plane and, moreover, we showed that $C'O$ must be an inertia line.

Thus B_1B_2' must be conjugate to $C'O$.

But B_1B_2' is conjugate to $C'D$, which lies in P as does also $C'O$ and therefore the inertia lines $C'O$ and $C'D$ must be identical; so that C' lies in OD.

It follows that E' must be identical with E and B_2' must be identical with B_2.

Further, C' must be identical with C and so B_1B_2 is conjugate to

both CA_1 and CA_2 and therefore is also conjugate to every inertia line in the inertia plane containing CA_1 and CA_2 which passes through C.

But this is the inertia plane which contains a and c, while the separation line b which lies in P, passes through O and is conjugate to c, must be parallel to B_1B_2, since B_1B_2 also lies in P.

Thus, by Theorem 100, b is conjugate to every inertia line in the inertia plane containing a and c which passes through the element O.

Thus the theorem is proved.

Remarks

All the postulates which have hitherto been introduced may be represented by ordinary geometric figures involving not more than three dimensions.

This may be done in the manner described in the introduction: the α and β sub-sets being represented by cones.

We have now however to introduce a new postulate which cannot be represented along with the others in a three-dimensional figure and which therefore gives our geometry a sort of four-dimensional character.

The new postulate is as follows:

Postulate XIX. **If P be any optical plane, there is at least one element which is neither before nor after any element of P.**

Since any element in an optical plane must lie in a generator, it will be *after* certain elements and *before* certain other elements of that optical plane.

It follows that any element such as is postulated in Post. XIX must lie outside P.

Again if P be an optical plane and A be any element which is neither *before* nor *after* any element of P, then an optical line through A parallel to any generator of P will be a neutral-parallel and accordingly any generator of an optical plane lies in at least one other distinct optical plane.

Since we already know that any optical line lies in at least one optical plane, it follows that *there are at least two distinct optical planes containing any optical line.*

This might be taken as an alternative form of the postulate.

If P and Q be two distinct optical planes having an optical line a in common, then any element of Q which does not lie in a must lie in a generator of Q, say b, which is a neutral-parallel of a.

Since any generator of P which is distinct from a is also a neutral-parallel of a, it follows by Theorem 28 that b is a neutral-parallel of every generator of P.

Since every element of P lies in a generator it follows that no element of Q lying outside a is either *before* or *after* any element of P.

Although Post. XIX is required in order to prove that there are at least two distinct optical planes containing any optical line, it is possible, without using this postulate, to prove that there are at least two distinct optical planes containing any separation line.

This may be done in the following manner:

Let b be the separation line and O be any element in it.

We already know that if we take any two inertia planes containing b, then b is conjugate to one single inertia line in each of them which passes through O.

If a_1 and a_2 be two such inertia lines, then, as was shown in Theorem 103, b is conjugate to every inertia line in the inertia plane containing a_1 and a_2 which passes through O.

Further, if c_1 and c_2 be the two generators of this inertia plane which pass through O it was also shown in the course of proving Theorem 103 that if we take any element O' of b distinct from O such element is neither *before* nor *after* any element of either c_1 or c_2.

Thus if we take an optical line through O' parallel to c_1 it will be a neutral-parallel and so b and c_1 lie in an optical plane.

Similarly b and c_2 lie in an optical plane.

These optical planes must be distinct since c_1 and c_2 are distinct optical lines which both pass through O.

THEOREM 131

If b be any separation line and O be any element in it, there are at least two inertia planes containing O and such that b is conjugate to every inertia line in each of them which passes through O.

Let Q be an optical plane containing b and let c be the generator of Q which passes through O.

Then by Post. XIX it follows, as we have already shown, that there is at least one other optical plane, say R, containing the optical line c.

Let d be any separation line in R and passing through O.

Then no element of d except O is either *before* or *after* any element of Q and O itself is neither *before* nor *after* any element of Q which lies outside c.

Thus no element of d is either *before* or *after* any element of b and so, by Theorem 107, there is at least one inertia line, say a, which is conjugate to both b and d.

Thus, as was shown in Theorem 128, a must be conjugate to every separation line which lies in the separation plane containing b and d and which passes through O.

Let S be the separation plane containing b and d, and let P be the inertia plane containing a and c.

Then since b is conjugate to a and since no element of b with the exception of O is either *before* or *after* any element of c it follows, by Theorem 104, that b is conjugate to every inertia line in P which passes through O.

Similarly, since d is conjugate to a and since no element of d with the exception of O is either *before* or *after* any element of c, it follows that d is conjugate to every inertia line in P which passes through O.

Thus any inertia line in P which passes through O is conjugate to both b and d and therefore is conjugate to every separation line passing through O and lying in the separation plane S.

It follows that P cannot have more than one element in common with S, for if it had, it would have a separation line in common with S and every inertia line in P which passed through O would require to be conjugate to one separation line lying in P, which is impossible.

Now by Theorem 129 there is one and only one separation line, say e, lying in S and passing through O which is conjugate to every inertia line passing through O and lying in the inertia plane containing a and b.

Let T be the inertia plane containing a and e.

Then, by Theorem 130, since b is conjugate to a, it follows that b is conjugate to every inertia line in T which passes through O.

But now b is conjugate to every inertia line lying either in T or P which passes through O and, since T contains the separation line e which lies in S while P does not contain any separation line in S, it follows that T and P are distinct inertia planes.

Thus the theorem is proved.

Remarks

It follows from this that if a separation line b have an element O in common with any inertia plane U and is conjugate to every inertia line in U which passes through O, then b is also conjugate to certain other inertia lines passing through O which do not lie in U.

It also follows directly that there are certain optical lines passing

through O, but not lying in U, which are such that no element of b with the exception of O is either *before* or *after* any element of them.

Another important point which arises in the last theorem is that we may have an inertia plane and a separation plane having only one element in common and such that each inertia line through the common element in the former is conjugate to every separation line through it in the latter.

THEOREM 132

If two distinct inertia planes P and P' have a separation line b in common and if another separation line c intersecting b in the element O be conjugate to every inertia line in P which passes through O, then if c be conjugate to one inertia line in P' which passes through O it is conjugate to every inertia line in P' which passes through O.

Let f_1 and f_2 be the two generators of P which pass through O and let D_1 be any element in f_1 which is *after* O.

Let the general line through D_1 parallel to b intersect f_2 in D_2.

Then $D_1 D_2$ is a separation line and so, since O is *before* D_1, it must also be *before* D_2.

Let E_1 be any element linearly between D_1 and D_2 and let E_2 be any element linearly between E_1 and D_2, while C is any element linearly between E_1 and E_2.

Then by Theorem 73 (a) OE_1 is an inertia line and E_1 is *after* O.

Similarly, since O is *before* both E_1 and D_2, it follows that OE_2 is an

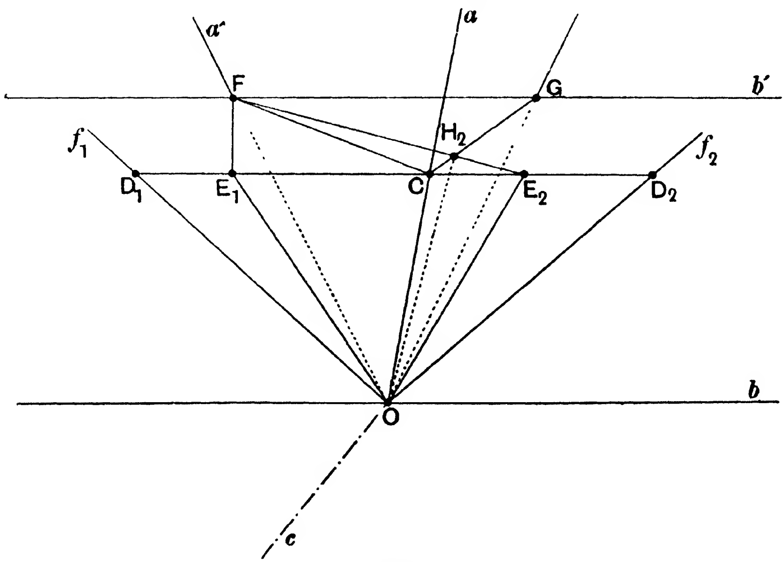


Fig. 38.

inertia line and E_2 is *after* O, and further since O is *before* both E_1 and E_2 it follows that OC must be an inertia line and C must be *after* O.

Thus OE_1, OE_2 and OC are three distinct inertia lines in P all passing through O and so the separation line c is conjugate to each of them.

Now if a' be an inertia line in P' which passes through O and to which c is conjugate, it follows by Theorem 103 that c is conjugate to every inertia line passing through O and lying in either of the three inertia planes containing a' and OE_1, a' and OE_2 or a' and OC.

Let F be any element of a' which is *after* O and let b' be the general line through F parallel to b.

Then b' must lie in P' and must be parallel to D_1D_2.

Let Q be the general plane containing b' and FC.

Then Q contains D_1D_2 and therefore also contains FE_1, FE_2 and FC.

Now any general line in P' which passes through O with the exception of b must intersect b' in some element, say G.

If now we consider the general line CG, we see that it must lie in Q since C and G are distinct elements in Q.

Further, CG must be distinct from E_1E_2 since E_1E_2 is parallel to b' while CG intersects b'.

Thus, since F, E_1 and E_2 do not lie in one general line while C is linearly between E_1 and E_2, it follows by Theorem 127 that, provided G does not coincide with F, the general line CG either intersects FE_2 in an element linearly between F and E_2 or else intersects FE_1 in an element linearly between F and E_1.

Consider the case where CG intersects FE_2 in an element H_2 linearly between F and E_2.

Then, since O is *before* both F and E_2, it follows by Theorem 73 that OH_2 is an inertia line, and since it lies in the inertia plane containing a' and OE_2 and passes through O, it follows that c is conjugate to it.

But c is also conjugate to OC and so, by Theorem 103, c is conjugate to every inertia line in the inertia plane containing OC and OH_2 which passes through O.

Similarly, if CG should intersect FE_1 in an element H_1 linearly between F and E_1, then c is conjugate to ever inertia line in the inertia plane containing OC and OH_1 which passes through O.

Thus in either case if OG should happen to be an inertia line, c must be conjugate to it.

Thus c must be conjugate to every inertia line in P' which passes through O and so the theorem is proved.

Remarks

Since, in the above theorem, there is one single inertia line through O in the inertia plane P which is conjugate to b, such inertia line will be conjugate to both b and c and so it follows, by Theorem 99, that no element of b is either *before* or *after* any element of c and so b and c must lie in a separation plane.

Again if f_1' and f_2' be the two generators of P' which pass through O, then no element of c with the exception of O will be either *before* or *after* any element of either f_1' or f_2'.

Now let a_1 be the one single inertia line through O and lying in P which is conjugate to b, and let a_1' be the one single inertia line through O and lying in P' which is conjugate to b.

Then a_1 and a_1' lie in an inertia plane, say R, and both b and c must be conjugate to every inertia line passing through O and lying in R.

Thus if g_1 and g_2 be the two generators of R which pass through O, no element of either b or c with the exception of O is either *before* or *after* any element of either g_1 or g_2.

Thus the optical lines g_1 and g_2 are such that g_1 and b lie in an optical plane and also g_2 and b lie in an optical plane.

The optical lines f_1' and f_2' on the other hand are such that both of them lie in an inertia plane containing b.

Normality of general lines having a common element

We are now in a position to define what we mean when we say that a general line a is "*normal*" to a general line b, which has an element in common with it.

Since a and b are not always general lines of the same kind, it is difficult to give any simple definition which will include all cases; but the introduction of the word "*normal*" is justified by the simplification which is thereby brought about in the statement of many theorems.

Only one case will be found to be strictly analogous to the normality of intersecting straight lines in ordinary geometry: namely the case of two separation lines.

The other cases are so different from our ordinary ideas of lines "*at right angles*" that we do not propose to use this expression in connexion with them.

Thus for instance any optical line is to be regarded as being "normal to itself", and the use of the words "at right angles" would, in this case, clearly be an abuse of language.

The extension of the idea of normality from the cases of general lines

having a common element to the cases of general lines which have not a common element is however quite analogous to the corresponding extension in ordinary geometry and will be made subsequently.

We are at present only concerned with the cases of general lines having a common element and shall naturally include among these that of an optical line being "normal to itself".

Thus the complete definition of the normality of general lines having a common element is to be taken as consisting of the following four particular definitions A, B, C and D.

Definition A. Any optical line will be said to be *normal to itself.*

Definition B. If an optical line a and a separation line b have an element O in common and if no element of b with the exception of O be either *before* or *after* any element of a, then b will be said to be *normal* to a, and a will be said to be *normal* to b.

Definition C. If an inertia line a and a separation line b be conjugate one to the other, then a will be said to be *normal* to b and b will be said to be *normal* to a.

Definition D. A separation line a having an element O in common with a separation line b will be said to be *normal* to b provided an inertia plane P exists containing b and such that every inertia line in P which passes through O is conjugate to a.

In this last case, since there is one single inertia line in P which passes through O and is conjugate to b, it is evident that a and b must lie in a separation plane.

If c be this inertia line then, by Theorem 130, every inertia line which passes through O and lies in the inertia plane containing c and a is conjugate to b and so b satisfies the definition of being normal to a.

Let the separation plane containing a and b be denoted by S.

Then c is conjugate to both a and b and therefore is conjugate to every separation line in S which passes through O.

It follows, by Theorem 129, that there is one and only one separation line in S and passing through O which is conjugate to every inertia line in P which passes through O and the separation line a has this property.

Now it is easy to see that a is the only separation line in S and passing through O which is normal to b; for suppose, if possible, that a' is another such separation line.

Then, by the definition, there must exist an inertia plane, say P', containing b and such that every inertia line in P' which passes through O is conjugate to a'.

Then there would exist one single inertia line, say c', through O and lying in P' which would be conjugate to b.

Thus c' would be conjugate to every separation line in S which passed through O and therefore would be conjugate to a.

But now P' could not be identical with P, for, as we have seen, a is the only separation line in S and passing through O which is conjugate to every inertia line in P which passes through O and a' has been supposed distinct from a.

But, by Theorem 132, it follows that a must be conjugate to every inertia line in P' which passes through O.

Thus we should have two distinct separation lines a and a' both lying in S and passing through O and both conjugate to every inertia line in P' which passes through O.

But this is impossible by Theorem 129, and so the assumption of the existence of two distinct separation lines in S which pass through O and are normal to b leads to a contradiction and therefore is not true.

Thus there is one and only one separation line in S which passes through O and is normal to b.

Again, since b lies in P while a cannot lie in P, it follows that if a separation line a be normal to a separation line b having an element in common with it, then a and b must be distinct.

If b be any general line in an inertia plane P and O be any element of b, then we know that if b be either an inertia or separation line there is one and only one general line through O and lying in P which is conjugate and therefore *normal* to b.

Also, from our definitions, if b be an optical line there is still one and only one general line through O and lying in P which is normal to b: namely b itself.

Thus we have the following general result:

If P be either a separation plane or an inertia plane and if b be any general line in P and O be any element in b, then there is one and only one general line lying in P and passing through O which is normal to b.

Now we have seen that if a separation line a be normal to a separation line b having an element in common with it, then a and b lie in a separation plane.

Thus two intersecting separation lines in an optical plane cannot be normal one to another.

Any separation line, however, which lies in an optical plane is normal to every optical line in the optical plane since no element of the separa-

tion line except the element of intersection is either *before* or *after* any element of any optical line in the optical plane.

Since there is one and only one optical line which passes through any element of an optical plane and lies in the optical plane we have the following result:

If P be an optical plane and if b be any separation line in P and O be any element in b, then there is one and only one general line lying in P and passing through O which is normal to b.

If on the other hand b be an optical line lying in P, then every general line in P which passes through O (including b itself) is normal to b.

We have now to prove the general theorem that: *if b and c be two distinct general lines having an element O in common and if a general line a passing through O be normal to both b and c, then a is normal to every general line which passes through O and lies in the general plane containing b and c.*

We have already proved a number of special cases of this general theorem.

(1) If b and c be both optical lines and a be a separation line, then b and c lie in an inertia plane, say P, and if O' be any element of a distinct from O there will be an inertia plane, say P', passing through O' and parallel to P.

Then O and O' will be representatives of one another in the parallel inertia planes P and P' and so, by Theorem 102, a is conjugate to every inertia line in P which passes through O.

Thus a is normal to every separation line in P which passes through O, to every inertia line in P which passes through O and to every optical line in P which passes through O.

(2) If b and c be both inertia lines and a be a separation line, the same result follows from Theorem 103.

(3) If b be an optical line and c an inertia line while a is a separation line, the same result follows from Theorem 104.

(4) If b be a separation line and c an inertia line while a is a separation line, it follows by Theorem 132 that a must be normal to every inertia line which passes through O and lies in the inertia plane containing b and c.

Thus as before, a must be normal to every general line which passes through O and lies in the inertia plane.

(5) If b and c be both separation lines and a an inertia line, then, as we have seen, b and c must lie in a separation plane and, as was shown in

Theorem 128, a is conjugate and therefore normal to every separation line passing through O and lying in this separation plane.

(6) If b be an optical line and c a separation line while a is identical with b, then, as we have already seen, b and c lie in an optical plane while a is normal to every general line which passes through O and lies in this optical plane.

Several other cases remain to be considered and these form the subject of Theorems 133 to 135.

We shall postpone the enumeration of the various remaining cases till we have proved these theorems.

Theorem 133

If a separation line c be normal to a separation line b which it intersects in the element O and if further c be normal to an optical line a' which it also intersects in the element O, then c is normal to every general line passing through O and lying in the general plane containing b and a'.

By the definition of normality there exists an inertia plane, say P, containing b and such that every inertia line in P which passes through O is conjugate to c.

In case a' should lie in this particular inertia plane the result follows directly and so we shall suppose that a' does not lie in P.

We shall therefore suppose that a' and b lie in a general plane P' distinct from P.

From the remarks at the end of Theorem 132 it is evident that P' may be either an inertia plane or an optical plane.

The mode of proof is similar to that employed in Theorem 132 except that a' is here an optical line instead of an inertia line.

Thus the proof that c is conjugate to every inertia line passing through O and lying in either of the three inertia planes containing a' and OE_1, a' and OE_2, or a' and OC, follows in this case from Theorem 104 instead of Theorem 103.

Everything else follows exactly as in Theorem 132 and we find that, if OG be any general line in P' which passes through O and is distinct from b, then OG lies in some inertia plane such that every inertia line in the latter which passes through O is conjugate to c.

Thus if OG be a separation line it satisfies the condition that c should be normal to it.

Also if OG should be either an optical line or an inertia line c must also be normal to it, and so the theorem is proved.

Remarks

From the definition of the normality of intersecting separation lines it is evident that we may have a separation line normal to two (or more) separation lines in an inertia plane.

From the last theorem it is also evident that we may have a separation line normal to two (or more) separation lines in an optical plane.

We may also have a separation line normal to two (or more) separation lines in a separation plane, as may easily be seen from the following considerations:

In the remarks at the end of Theorem 131 it was pointed out that we may have an inertia plane and a separation plane having only one element in common and such that each inertia line through the common element in the former is conjugate to every separation line through it in the latter.

Let P be the inertia plane, S the separation plane and O the common element.

Let a and b be any two separation lines passing through O and lying in S, and let c be any separation line passing through O and lying in P.

Then a satisfies the definition of being normal to c and therefore c is normal to a.

Similarly c must be normal to b.

Thus c is normal to the two separation lines a and b which lie in the separation plane S.

Theorem 134

If three distinct separation lines a, b and c have an element O in common and if c be normal to both a and b, then c is normal to every general line which passes through O and lies in the general plane containing a and b.

By the definition of the normality of intersecting separation lines there must exist an inertia plane, say P, containing b and such that every inertia line in P which passes through O is conjugate to c.

Let f_1 and f_2 be the two generators of P which pass through O and let D_1 be any element in f_1 which is *after* O.

Let the separation line through D_1 parallel to b intersect f_2 in D_2.

Then D_2 must also be *after* O.

Let C be any element linearly between D_1 and D_2.

Then by Theorem 73 OC is an inertia line and C is *after* O.

But c is normal to the inertia line OC and to the separation line a and therefore by case (4) on p. 212 c must be normal to every inertia line

(and therefore also every general line) which passes through O and lies in the inertia plane containing OC and a.

Let R be this inertia plane.

If R should coincide with P the result follows directly and so we shall suppose that R is distinct from P.

Let S be the general plane containing a and b.

Then S will be distinct from both P and R, and, as was pointed out in the remarks at the end of the last theorem, S may be an inertia plane, an optical plane, or a separation plane.

Let one of the generators of R which pass through C intersect a in G_0 and let the generator of the opposite set passing through O intersect CG_0 in F.

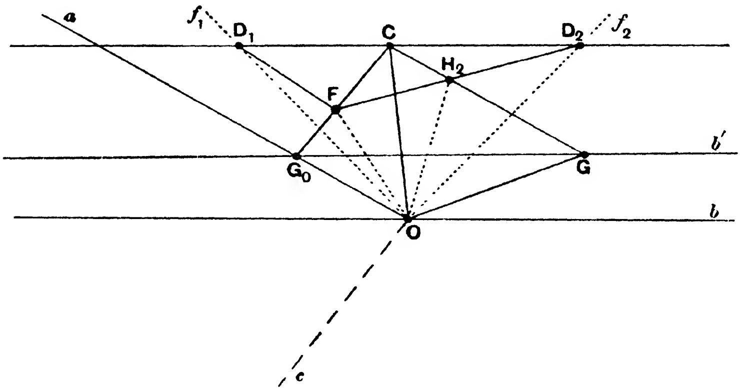


Fig. 39.

Then, since O does not lie in the optical line CG_0 but is *before* the element C of it, it follows that F must lie in the α sub-set of O and therefore F is *after* O.

Let b' be the general line through G_0 parallel to b.

Then, since G_0 lies in S, it follows that b' lies in S.

Let Q be the general plane containing b' and G_0C.

Then, since D_1D_2 is parallel to b and is distinct from b', it follows that it is parallel to b' and, since D_1D_2 passes through C, it must lie in the general plane Q.

Thus D_1, D_2 and F are three distinct elements in Q which do not all lie in one general line.

Now any general line in S which passes through O and is distinct from b must intersect b' in some element, say G.

Then the general line CG lies in Q and is distinct from D_1D_2.

If then G does not coincide with G_0 it follows, by Theorem 127, that

CG must either intersect D_1F in an element H_1 linearly between D_1 and F, or else must intersect D_2F in an element H_2 linearly between D_2 and F.

But, since O is *before* both D_1 and F, it follows, by Theorem 73, that OH_1 is an inertia line and similarly, since O is *before* both D_2 and F, it follows that OH_2 is an inertia line.

Now c is normal to every general line in P which passes through O and also to every general line in R which passes through O and therefore c is normal to the three optical lines OD_1, OD_2 and OF.

Thus c must be conjugate to every inertia line which passes through O and lies either in the inertia plane containing OD_1 and OF, or the inertia plane containing OD_2 and OF.

Thus c is conjugate to OH_1 and also to OH_2.

But c is conjugate to OC and therefore is conjugate to every inertia line which passes through O and lies in the inertia plane containing OC and OH_1 or the inertia plane containing OC and OH_2.

Thus, since OG lies in the inertia plane containing OC and OH_1 or in the inertia plane containing OC and OH_2 as the case may be, it follows that c must be normal to OG.

Thus, including the separation lines a and b, the separation line c is normal to every general line which passes through O and lies in the general plane S.

THEOREM 135

If two distinct separation lines a and b intersect in an element O and if an optical line c passing through O be normal to both a and b, then c is normal to every general line which passses through O and lies in the general plane containing a and b.

From the definition of the normality of an optical line to an intersecting separation line it follows that c and a lie in an optical plane, say P, while c and b lie in an optical plane, say Q.

If P should be identical with Q we already know that c is normal to every general line in P which passes through O including the optical line c itself.

Let us suppose next that P is distinct from Q.

We have already seen that in this case a and b lie in a separation plane, say S, and further we have seen that no element of b with the exception of O is either *before* or *after* any element of P.

Let D be any element of b distinct from O and let E be any element of a distinct from O, while F is any element of a such that O is linearly between E and F.

Let e and f be optical lines through E and F respectively and parallel to c.

Then D is neither *before* nor *after* any element either of e or of f and so, by Theorem 45, no element of DE with the exception of E is either *before* or *after* any element of e, and no element of DF with the exception of F is either *before* or *after* any element of f.

But now by Theorem 127 any general line passing through O and lying in S and which is distinct from both a and b must either intersect

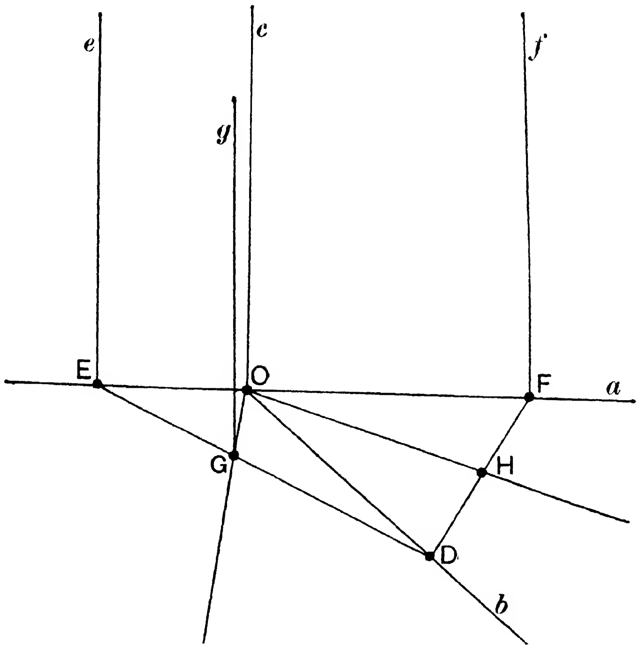


Fig. 40.

DE in some element, say G, linearly between D and E or else must intersect DF in some element, say H, linearly between D and F.

Thus G is neither *before* nor *after* any element of e while H is neither *before* nor *after* any element of f.

If then g be an optical line through G parallel to e it will be a neutral-parallel and, since c is a neutral-parallel of e and G does not lie in c, it follows by Theorem 28 that g is a neutral-parallel of c.

Thus G is neither *before* nor *after* any element of c and therefore, by Theorem 45, no element of OG with the exception of O is either *before* or *after* any element of c.

Thus c is normal to OG and similarly it is normal to OH.

It follows that c is normal to every general line which passes through O and lies in S, and so the theorem is proved.

Enumeration of cases of general theorem continued

We now resume the enumeration of the various cases of the general theorem stated on p. 212 and which was interrupted in order to prove Theorems 133 to 135.

Six cases have already been mentioned and we now proceed with case (7).

(7) If b be a separation line and c an optical line while a is a separation line and if b and c lie in an inertia plane, the result follows from Theorem 133.

(8) If b and c be both separation lines lying in an inertia plane and if a be also a separation line, the result follows from Theorem 134.

(9) If b be a separation line and c an optical line while a is a separation line and if b and c lie in an optical plane, the result follows from Theorem 133.

(10) If b and c be both separation lines lying in an optical plane and if a be also a separation line, the result follows from Theorem 134.

(11) If b and c be both separation lines lying in an optical plane and if a be an optical line also in the optical plane, the result still holds as was pointed out at the beginning of Theorem 135.

(12) If b and c be both separation lines lying in a separation plane and if a be also a separation line, the result follows from Theorem 134.

(13) If b and c be both separation lines lying in a separation plane and if a be an optical line, the result follows from Theorem 135.

If now we combine cases (1), (2), (3), (4), (7) and (8) we see that b and c may be any two intersecting general lines in an inertia plane taking a as a separation line.

If we combine cases (9) and (10) we see that b and c may be any two intersecting general lines in an optical plane taking a as a separation line.

Further, combining cases (6) and (11) we also see that b and c may be any two intersecting general lines in an optical plane taking a as an optical line.

Finally from cases (12), (13) and (5) we see that b and c may be any two intersecting general lines in a separation plane taking a as a separation line, an optical line, or an inertia line.

Thus for all the different possible cases of the normality of general lines having a common element this general result holds.

Theorem 136

If b and c be two separation lines intersecting in an element O and lying in a separation plane S and such that c is normal to b, then if O' be any other element of b, the normal to b through O' in the separation plane S is parallel to c.

From the definition of the normality of intersecting separation lines it follows that there must exist an inertia plane P containing b and such that every inertia line in P which passes through O is conjugate to c.

Let a_1 and a_2 be any two such inertia lines and let a_1' and a_2' be inertia lines passing through O' and parallel to a_1 and a_2 respectively.

Let c' be a separation line passing through O' and parallel to c.

Then c' will lie in S.

But, by Theorem 100, both a_1' and a_2' must be conjugate to c' and so, by Theorem 103, c' is conjugate to every inertia line in the inertia plane containing a_1' and a_2' which passes through the element O'.

But this inertia plane is the inertia plane P which contains the separation line b and so c' satisfies the definition of being normal to b.

Further, c' passes through O' and lies in S and we have already seen that there is only one normal to b which satisfies these conditions.

Thus the normal to b through O' in the separation plane S is parallel to c as was to be proved.

Theorem 137

If b and c be two separation lines intersecting in an element O and such that c is normal to b and if b' and c' be two other separation lines intersecting in an element O' and respectively parallel to b and c, then c' is normal to b'.

Since c is normal to b there must exist an inertia plane P containing b and such that every inertia line in P which passes through O is conjugate to c.

Let a_1 be one such inertia line which we shall suppose does not also pass through O'.

Then through O' there is an inertia line, say a_1', which is parallel to a_1.

Thus b' and a_1' determine an inertia plane P' which will be either identical with P or parallel to P according as O' does or does not lie in P.

Let a_2 be a second inertia line in P and passing through O but not through O'.

Then through O' there is an inertia line say a_2' parallel to a_2 and lying in P'.

Then by Theorem 100 both a_1' and a_2' are conjugate to c' and so, by Theorem 103, c' is conjugate to every inertia line in P' which passes through O'.

But P' contains b' and so c' satisfies the definition of being normal to b'.

Theorem 138

If an optical line b intersects a separation line c in an element O and if c be normal to b and if further b' and c' be an optical line and a separation line respectively which intersect in an element O' and are respectively parallel to b and c, then c' will be normal to b'.

From the definition of the normality of a separation line to an optical line it follows that b and c lie in an optical plane, say P.

Further, b' and c' lie in a general plane P' which must be either identical with P or parallel to P according as O' does or does not lie in P.

In either case P' is an optical plane and accordingly, since b' is an optical line and c' a separation line, it follows that c' must be normal to b'.

Remarks

By combining Theorems 100, 137 and 138 we obtain the general result that *if b and c be two general lines intersecting in an element O and such that the one is normal to the other and if b' and c' be two other general lines intersecting in an element O' and respectively parallel to b and c, then of these latter two general lines the one is normal to the other.*

If now we remember that an optical line is to be regarded as *normal to itself*, we are in a position to extend the definition of the normality of general lines to the case of general lines which have no element in common, as is done with straight lines in ordinary geometry.

Definition. A general line b will be said to be *normal* to a general line c' which has no element in common with it, provided that a general line b' taken through any element of c' parallel to b is normal to c' in the sense already defined.

It is evident from the above considerations that, in these circumstances, if a general line c be taken through any element of b parallel to c' then c will be normal to b and so c' will be normal to b.

Further, we have the result that *any two parallel optical lines are to be regarded as normal to one another.*

Again, if P be an inertia plane and if a be any general line in P and A be any element in P, then there is one single general line in P and passing through A which is normal to a.

If however a be an optical line, the normal to a through A is either identical with a or parallel to it according as A does or does not lie in a.

If, on the other hand, P be an optical plane, there is one single general line in P and passing through A which is normal to a, except when a is an optical line, in which case every general line in P which passes through A is normal to a.

If P be a separation plane there is one single general line in P which passes through A and is normal to a and in this case the normal to a always intersects a as in ordinary geometry.

Definition. A general line a will be said to be *normal* to a general plane P provided a be normal to every general line in P.

It is evident that if a general line a be normal to two intersecting general lines in a general plane P, then a will be normal to P.

In case P be an optical plane it is clear that, according to the above definition, any generator of P is normal to P.

This is the only case in which a general line can be normal to a general plane which contains it.

In no other case can a general line which is normal to a general plane have more than one element in common with the latter.

As was pointed out in the remarks at the end of Theorem 131 we may have an inertia plane and a separation plane having only one element in common and such that each inertia line through the common element in the former is conjugate to every separation line through it in the latter.

It is evident now that we have here two general planes which are so related that any general line in the one is normal to any general line in the other.

In ordinary *three-dimensional* geometry two planes cannot be so related, and when we speak of one plane being normal to another the normality is not of this complete character.

We shall therefore introduce the following definition:

Definition. If two general planes be so related that every general line in the one is normal to every general line in the other, the two general planes will be said to be *completely normal* to one another.

Theorem 139

If P be an inertia plane and O be any element in it, there is at least one separation plane passing through O and completely normal to P.

Let P_1 be any inertia plane which is parallel to P and let O_1 be the representative of O in P_1.

Then, by Theorem 102, the separation line OO_1 is conjugate to every inertia line in P which passes through O and so OO_1 is normal to P.

Let a_1 be one of the two generators of P which pass through O, and let OO_1 be denoted by b_1.

Then a_1 and b_1 lie in an optical plane, say Q_1.

Now, by Post. XIX, there is at least one element, say A, which is neither *before* nor *after* any element of Q_1.

Thus through A there is an optical line, say a_1', which is neutrally parallel to a_1 and so a_1 and a_1' lie in an optical plane, say R_1, which is distinct from Q_1.

Again if P_2 be an inertia plane through A parallel to P it will contain a_1'.

Let O_2 be the representative of O in P_2.

Then O_2 must lie in a_1' and so OO_2 must lie in the optical plane R_1.

But OO_1 lies in Q_1 while OO_2 lies in R_1, and Q_1 and R_1 have only the optical line a_1 in common.

Thus since OO_1 and OO_2 are both separation lines they must be distinct.

Now, by Theorem 102, OO_2 is conjugate to every inertia line in P which passes through O, and so OO_2 is normal to P.

Let OO_2 be denoted by b_2.

Then no element of b_2 is either *before* or *after* any element of b_1 and, since b_1 and b_2 have the element O in common, they must lie in a separation plane, say S.

Thus any inertia line in P which passes through O is conjugate to both b_1 and b_2 and therefore also conjugate to every separation line in S which passes through O.

Thus every general line in P is normal to every general line in S and so the separation plane S is completely normal to P.

Thus, since S passes through O, the theorem is proved.

Theorem 140

If P be a separation plane and O be any element in it, there is at least one inertia plane passing through O and completely normal to P.

If we take any two separation lines in P and passing through O then, by Theorem 107, there is at least one inertia line, say a_1, which is conjugate to both of them and therefore is normal to P.

Let b_1 be any separation line in P which passes through O and let Q be the inertia plane containing a_1 and b_1.

Then, by Theorem 129, there is one and only one separation line in P and passing through O which is conjugate to every inertia line in Q which passes through O.

Let b_2 be this separation line.

Then, as was remarked at the end of Theorem 131, b_2 is conjugate to certain other inertia lines passing through O which do not lie in Q.

Let a' be any such inertia line and let Q' be the inertia plane containing a' and b_1.

Then, by Theorem 132, b_2 is conjugate to every inertia line in Q' which passes through O.

Let a_2 be the one single inertia line in Q' and passing through O which is conjugate to b_1 and let R be the inertia plane containing a_1 and a_2.

Then a_1 and a_2 are each conjugate to both b_1 and b_2.

Thus both a_1 and a_2 are conjugate to every separation line in P which passes through O and so every separation line in P which passes through O is conjugate to every inertia line in R which passes through O.

Thus every general line in P is normal to every general line in R and so the inertia plane R is completely normal to P.

Thus, since R passes through O, the theorem is proved.

Theorem 141

If P be an optical plane and O be any element in it, there is at least one optical plane passing through O and completely normal to P.

Let a be the generator of P which passes through O and let b be any separation line in P which passes through O.

Then, by Post. XIX, there is at least one element, say A, which is neither *before* nor *after* any element of P.

The general line OA is thus a separation line and, by Theorem 45, no element of OA with the exception of O is either *before* or *after* any element of a.

Thus a is normal to OA and it is also normal to b and so, since OA and

b must lie in a separation plane, say S, it follows that the optical line a is normal to S.

But now we know that there is one single separation line, say c, which passes through O, lies in S and is normal to b.

Then c is normal to both a and b and therefore is normal to P.

But c and a lie in an optical plane which is distinct from P and which we shall call R.

Further, a is an optical line in P and therefore is normal to P.

Thus any general line in P is normal to the two intersecting general lines a and c which lie in R and so every general line in P is normal to every general line in R.

It follows that R is completely normal to P and, since R passes through O, the theorem is proved.

Remarks

By combining Theorems 139, 140 and 141 we get the general result:

If P be any general plane and O be any element in it, there is at least one general plane passing through O and completely normal to P.

If R be this general plane which is completely normal to P and if O' be any element not lying in P, then O' either may or may not lie in R.

If O' does not lie in R, then there is a general plane, say R', passing through O' and parallel to R.

It is evident that since R is completely normal to P we must also have R' completely normal to P and so we may generalise the above result and say:

If P be any general plane and O be any element whatever, there is at least one general plane passing through O and completely normal to P.

Let O be any element and let S be any separation plane passing through O, while P is an inertia plane also passing through O and completely normal to S.

Let a be any separation line in S which passes through O and let b be the one single separation line in S passing through O which is normal to a.

Let c be any separation line passing through O and lying in P and let d be the one single inertia line in P and passing through O which is normal to c.

Then both c and d are normal to both a and b and so *we have the three separation lines a, b and c all passing through O and each of them normal to the other two; while in addition to these we have the inertia line d also passing through O and normal to all three.*

This result marks an important stage in the development of our theory, as it suggests the possibility of setting up a system of normal coordinate axes one of which axes is of a different character from the remaining three.

Another important result is the following:

If S be a separation plane and if P be an inertia plane passing through any element O of S and completely normal to S, then there are two generators of P which pass through O and each of them is normal to the separation plane S.

Thus there are at least two optical lines which pass through any element of a separation plane and are normal to it.

THEOREM 142

If P be an inertia or separation plane and O be any element which does not lie in it, there is one single general line passing through O and normal to P which has an element in common with P.

We already know that if a be a separation line and if O be any element which does not lie in it, then, in whatever type of general plane O and a may lie, there is one single general line passing through O and lying in this general plane which is normal to a.

Further, if d be this general line normal to a, then d must intersect a in some element, say A.

Now suppose that a lies in the inertia or separation plane P.

Then there is one single general line passing through A and lying in P which is normal to a.

Let b be this general line.

Then, since P is an inertia or separation plane and a is a separation line, b must be distinct from a and must be either an inertia or separation line and cannot be an optical line.

Now we know that in whatever type of general plane O and b may lie there is one single general line passing through O and lying in this general plane which is normal to b.

Let c be this general line.

Then, since b is not an optical line, this normal to it through O cannot be parallel to b and therefore must intersect b in some element, say B.

Now a is normal to the two general lines d and b which intersect in A and accordingly a is normal to every general line in the general plane containing d and b and therefore is normal to c.

But c is normal to the two intersecting general lines a and b which lie in P and therefore c is normal to P.

Since c has the element B in common with P, we have proved that there is at least one general line through O and normal to P which has an element in common with P.

It remains to show that there is only one general line having this property.

Consider first the case where P is a separation plane and let B' be any element in P distinct from B.

Then BB' is a separation line and so in whatever type of general plane O and BB' may lie there is one single general line passing through O, lying in this general plane and normal to BB'.

But OB passes through O and is normal to BB' and therefore OB' cannot be normal to BB' and so cannot be normal to P.

This proves that OB is the only general line through O and normal to P which has an element in common with P provided P be a separation plane.

This method does not serve if P be an inertia plane, since BB' might, in this case, be an optical line.

If P be an inertia plane, let P' be an inertia plane passing through O and parallel to P.

Then O and B must be representatives of one another in the parallel inertia planes P' and P.

If B' be any other element in P distinct from B and we suppose that OB' is normal to P, then B' would also be the representative of O in P, which we know is impossible.

Thus again OB is the only general line through O and normal to P which has an element in common with P.

The theorem thus holds for both separation and inertia planes.

Theorem 143

If P be an optical plane and O be any element which does not lie in it, then:

(1) *If O be neither* before *nor* after *any element of P there is one single generator of P such that every general line which passes through O and intersects this generator is normal to P.*

(2) *If O be either* before *or* after *any element of P there is no general line passing through O and having an element in common with P which is normal to P.*

As regards the first part of this theorem, if we carry out the construction of Theorem 142 taking a as a separation line, then, since P is

an optical plane, the general line b must be an optical line since it is normal to a.

Since O is neither *before* nor *after* any element of P, it is neither *before* nor *after* any element of b.

If then OB be any general line passing through O and intersecting b in the element B, it follows by Theorem 45 that no element of OB with the exception of B is either *before* or *after* any element of b.

It follows that OB is normal to b.

But, as in Theorem 142, OB is normal to a and thus OB is normal to the two intersecting general lines a and b which lie in P and therefore it is normal to P.

Again if B' be any element in P which does not lie in b, then BB' is a separation line and so, as in Theorem 142, OB' cannot be normal to P.

Thus all general lines through O which have an element in common with b are normal to P, and no other general line through O which intersects P can be normal to P.

Thus the first part of the theorem is proved.

Suppose next that O is *before* some element, say E, in P.

Then through E one single generator of P passes which we may denote by f.

Since O does not lie in f but is *before* an element of f, it follows that through O there is an optical line which is a before-parallel of f and which we shall denote by c.

If f' be any other generator of P it will be a neutral-parallel of f and so by Theorem 26 (a) c will be a before-parallel of f'.

Thus O is *before* elements of every generator of P.

Similarly if O be *after* any element of P it is *after* elements of every generator of P.

Thus in case O be either *before* or *after* any element of P it will lie in an inertia plane along with any selected generator of P.

Let OB be any general line passing through O and having the element B in common with P and let b be the generator of P which passes through B.

Then OB and b lie in an inertia plane and intersect in B and so, since b is an optical line, OB cannot be normal to it.

Thus OB cannot be normal to P and therefore there is in this case no general line passing through O and having an element in common with P which is normal to P.

THEOREM 144

If a general line d have an element A in common with a general plane P, there is at least one general line passing through A and lying in P which is normal to d.

If d lies completely in P we already know that the theorem holds and so we shall suppose that A is the only element common to d and P.

We shall first consider the case where P is an inertia or separation plane.

In this case, if O be any element of d distinct from A, there is, by Theorem 142, one single general line passing through O and normal to P which has an element in common with P.

Let B be this element.

If B should coincide with A, then every general line passing through A and lying in P would be normal to d.

If B does not coincide with A let a be the one single general line passing through A and lying in P which is normal to AB.

Then since OB is normal to P it must be normal to a.

Thus a is normal to the two intersecting general lines AB and OB and therefore is normal to the general plane containing them.

Thus the general line a must be normal to d and, since a passes through A and lies in P, the theorem is proved for the case where P is an inertia or separation plane.

Suppose next that P is an optical plane and let b be the generator of P which passes through A.

Now, since b is an optical line, it follows that the intersecting general lines b and d must lie in a general plane, say Q, which must be either an optical plane or an inertia plane.

Suppose first that Q is an optical plane.

Then, since b is an optical line in Q and d intersects b, it follows that d must be a separation line and b must be normal to d.

But b passes through A and lies in P and so the theorem is proved for this case.

Next consider the case where Q is an inertia plane.

Let b' be any generator of P distinct from b.

Then, since b' is a neutral-parallel of b, it follows that an inertia plane Q' through any element of b' and parallel to Q will contain b'.

If then A' be the representative of A in Q', the general line AA' will be normal to Q and therefore will be normal to d.

But, since b' is neutrally parallel to b which contains the element A, the element A' must lie in b' and therefore in the optical plane P.

Thus the general line AA' must lie in P and, since it passes through A and is normal to d, the theorem holds also in this case.

Thus the theorem holds in general.

Theorem 145

If three general lines a, b and c have an element O in common, there is at least one general line passing through O which is normal to all three.

If we take any two of the three given general lines, say a and b, it follows, since they have the element O in common, that they lie in a general plane, say P.

Then by Theorems 139, 140 and 141 there is at least one general plane passing through O and completely normal to P.

Let Q be this general plane.

Then, since c has the element O in common with Q, it follows, by Theorem 144, that there is at least one general line, say d, passing through O and lying in Q which is normal to c.

But, since d lies in Q, it is normal to both a and b and thus is normal to all three general lines.

Thus the theorem is proved.

Definition. If a general line and a general plane have one single element in common, they will be said to *intersect* in that element.

Definition. If a general line a and a general plane P intersect, then the aggregate of all elements of P and of all general planes parallel to P which intersect a will be called a *general threefold.*

It will be found that, just as there are three types of general line and three types of general plane, so there are three types of general threefold.

In the case of general threefolds, however, unlike that of general lines or of general planes, we are able to give a definition which applies to all three types without first considering any of the special cases.

From the definition it is clear that if a general threefold W be determined by a general line a intersecting a general plane P, then any other general plane P_1 parallel to P and intersecting a may take the place of P, so that a and P_1 will also serve to determine W.

Again if a intersects P in the element O and if a' be a general line parallel to a and intersecting P in another element O', then a and a' will lie in a general plane, say Q.

If through any element O_1 of a distinct from O the general plane P_1 passes parallel to P, then, by Theorem 123, the general plane Q must have a second element in common with P_1.

Thus P_1 and Q have a general line in common which must be parallel to OO' and so the general line a' must intersect P_1 in some element O_1'.

Thus a' intersects every general plane parallel to P which intersects a, and similarly, a intersects every general plane parallel to P which intersects a'.

It follows that every element of a' lies in the general threefold determined by a and P, and also: that a' and P determine the same general threefold as a and P.

Theorem 146

If two distinct elements of a general line lie in a general threefold, then every element of the general line lies in the general threefold.

Let the general threefold W be determined by a general plane P and a general line a which intersects it.

Let X_1 and X_2 be two distinct elements of a general line b and let them both lie in W.

If X_1 and X_2 should both lie in P or in any one of the general planes which intersect a and are parallel to P, then the general line b will lie in that general plane and therefore every element of b must lie in W.

We shall next suppose that X_1 lies in one of the set of parallel general planes, say P_1, while X_2 lies in another, say P_2.

Then b either may or may not lie in a general plane containing a.

Suppose first that b lies in a general plane Q along with a.

Then we may have either:

(1) b identical with a,

or (2) b parallel to a,

or (3) b intersecting a.

If b be identical with a the result is obvious.

If b be parallel to a then, as we have already shown, every element of b lies in W.

If b intersects a, then at least one of the elements X_1, X_2 must be distinct from the element of intersection of b and a.

We may suppose that X_1 is distinct from this element of intersection.

Then the element in which a intersects P_1 must be distinct from X_1 and so the general plane Q has two distinct elements in common with P_1.

Further, since the general line a intersects all the general planes parallel to P_1 whose elements along with the elements P_1 make up W, it follows, by Theorem 123, that Q has a general line in common with each of these general planes and all these general lines are parallel to one another.

Now since b does not lie in P_1 it follows that b must intersect all these general planes, and similarly a general plane through any element of b distinct from X_1 and taken parallel to P_1 must intersect a.

Thus we see that in this case also every element of b lies in W and further that b and P_1 determine the same general threefold as a and P_1: namely W.

Thus the theorem holds provided b and a lie in one general plane.

Finally suppose as before that X_1 lies in P_1 and X_2 in P_2 and that b and a do not lie in one general plane.

Let a intersect P_1 in the element Y_1 and let b' be a general line through Y_1 parallel to b.

Then b and b' lie in a general plane, say R, which has the two elements X_1 and Y_1 in common with P_1 and has the element X_2 in common with the parallel general plane P_2.

Thus R has a general line in common with P_2 which is parallel to X_1Y_1 and so b' must intersect P_2 in some element, say Y_2.

But now, from what we have already proved, every element of b' must lie in W and also b' and P_2 determine the general threefold W equally with a and P_2 or a and P.

Again, since b is parallel to b', it follows from what we have already proved that every element of b lies in the general threefold determined by b' and P_2: that is in W; and that b and P_2 may also be taken as determining the general threefold W.

Thus the theorem holds in general.

Remarks

It is evident from the above that if a general threefold W be determined by a general plane P and a general line a which intersects P, then a general line b which has two distinct elements in common with W, which do not both lie in P or do not both lie in one of the general planes parallel to P and intersecting a, will intersect all these general planes including P.

Further, b and P, or b and any one of these general planes, will also determine W.

Again if a general plane Q have two distinct elements X_1 and X_2 in

common with W, then Q will have at least one general line in common with W: namely the general line X_1X_2 since, by the above theorem, every element of X_1X_2 must lie in W, and we already know that every element of it must also lie in Q.

It is not however possible from this to prove that Q and W have more than one general line in common.

Theorem 147

If a general plane have three distinct elements in common with a general threefold and if these three elements do not all lie in one general line, then every element of the general plane lies in the general threefold.

Let the general threefold W be determined by a general plane P and a general line a which intersects P.

Let X_1, X_2 and X_3 be three distinct elements of a general plane Q which do not all lie in one general line and suppose that X_1, X_2 and X_3 all lie in W.

If all these three elements should lie in P or if they should all lie in one of the general planes parallel to P which intersect a, then Q would be identical with the general plane in which they all lie and accordingly every element of Q would lie in W.

If X_1, X_2 and X_3 do not all lie in one of this set of general planes, suppose that X_1 lies in the general plane P_1 of the set while X_2 lies in another distinct general plane of the set, say P_2.

Then X_3 will lie in some general plane P_3 of the set which may be either identical with P_1 or with P_2, or may be distinct from both.

Now, since X_1 and X_2 lie in two distinct general planes of the set, it follows that the general line X_1X_2 intersects every general plane of the set and therefore must intersect P_3 in some element, say O.

Further, since X_1, X_2 and X_3 do not all lie in one general line, it follows that X_3 and O must be distinct elements.

Thus the general planes P_3 and Q have two distinct elements X_3 and O in common and therefore have the general line OX_3 in common, which accordingly lies in W.

Again the general threefold W, as we have seen, may be determined by the general plane P_3 and the general line X_1X_2 which intersects P_3 in O.

But now every element of Q lies either in X_1X_2 or in a general line parallel to X_1X_2 and intersecting OX_3.

We have however seen that every element of any such general line must lie in W.

It follows that every element of Q must lie in W.

Thus the theorem holds in all cases.

THEOREM 148

(1) *If a general line b lies in a general threefold W and if A be any element lying in W but not in b, then the general line through A parallel to b also lies in W.*

(2) *If a general plane P lies in a general threefold W and if A be any element lying in W but not in P, then the general plane through A parallel to P also lies in W.*

The first part of the theorem may be proved as follows:

The general line b and the element A determine a general plane, say Q, having three elements in common with W which do not all lie in one general line, and so, by Theorem 147, Q lies in W.

But the general line through A parallel to b must lie in Q and therefore must lie in W.

This proves the first part of the theorem.

In order to prove the second part let b and c be two intersecting general lines which both lie in P and therefore in W.

The element A does not lie in P and therefore cannot lie either in b or c.

If then b' and c' be general lines through A parallel to b and c respectively, it follows from the first part of the theorem that b' and c' both lie in W.

If then P' be the general plane containing b' and c' it will contain three distinct elements in common with W which do not all lie in one general line and so, by Theorem 147, P' must lie in W.

But P' is parallel to P and passes through A and so the theorem is proved.

THEOREM 149

If a general threefold W be determined by a general plane P and a general line a which intersects P, then if Q be any general plane lying in W, and if b be any general line lying in W and intersecting Q, the general plane Q and the general line b also determine the same general threefold W.

It is evident from the remarks at the end of Theorem 146 that the above holds in the special case where Q is one of the set of general planes consisting of P and all general planes parallel to P which intersect a.

We shall therefore consider the case where Q is distinct from any

one of this set of general planes which we shall for convenience refer to as the *primary set*.

Let X_1 be any element in Q and take any two distinct general lines lying in Q and passing through X_1.

Then these could not both lie in any general plane of the primary set, for if so Q would require to be identical with that general plane, contrary to hypothesis.

Thus at least one of the two general lines does not lie in any general plane of the primary set.

Suppose c_1 be a general line of this character.

Then, since Q lies in W, each element of c_1 must lie in a distinct general plane of the primary set, and c_1 must intersect every general plane of the primary set.

Thus W may be determined by any general plane of the primary set and the general line c_1 which intersects it, in place of the general line a.

Let X_2 be any element of Q which does not lie in c_1 and let c_2 be a general line through X_2 parallel to c_1.

Then c_2 must also lie in Q and must also intersect every general plane of the primary set.

Further, since c_1 and c_2 are parallel, they must intersect any general plane of the primary set in distinct elements, and accordingly any general plane of the primary set has a general line in common with Q.

Now let B be any element in b other than its element of intersection with Q.

Then B must lie in some general plane of the primary set, say P_1, since b lies in W.

Now, as we have seen, P_1 has a general line in common with Q and, since B does not lie in Q, it cannot lie in this general line.

If C and D be any two distinct elements in this general line, then B, C and D are three distinct elements in P_1 which do not all lie in one general line.

But now, if W' be the general threefold determined by Q and b, it is evident that B, C and D lie in W' and so, by Theorem 147, the general plane P_1 must lie in W'.

Also since c_1 lies in Q it must lie in W', and so, by Theorem 148, every general plane which passes through an element of c_1 and is parallel to P_1 must lie in W'.

But the general threefold W is the aggregate of all elements of P_1 and of all general planes parallel to P_1 which intersect c_1, and so every element of W must lie in W'.

But, since Q and b both lie in W, it follows by Theorem 148 that every general plane which passes through an element of b and is parallel to Q must lie in W.

Since, however, the general threefold W' is the aggregate of all elements of Q and of all general planes parallel to Q which intersect b, it follows that every element of W' must lie in W.

Thus the general threefolds W' and W consist of the same set of elements and are therefore identical.

Thus Q and b determine W, as was to be proved.

Remarks

It follows directly from the above theorem that *any four distinct elements which do not all lie in one general plane determine a general threefold containing them.*

For let A, B, C, D be four distinct elements which do not all lie in one general plane.

Then no three of them can lie in one general line.

Let Q be the general plane containing A, B and C and let b be the general line DA.

Then b cannot have any other element than A in common with Q, for then D would have to lie in Q along with A, B and C contrary to hypothesis.

Thus b intersects Q.

Let W be the general threefold determined by Q and b and let W' be any general threefold containing A, B, C and D.

Then, since W' contains A, B and C, it follows by Theorem 147 that W' contains Q.

Also by Theorem 146 since W' contains A and D it contains b.

Thus by Theorem 149 the general threefold W' is identical with W: that is to say is identical with one definite general threefold.

Again it is clear that: *any three distinct general lines having a common element and not all lying in one general plane determine a general threefold containing them.*

Theorem 150

If two distinct general planes P and Q lie in a general threefold W, then if P and Q have one element in common they have a second element in common.

Let A be any element in P and let B be any element which lies in W but not in P.

Let the general line AB be denoted by a.

Then a intersects P and, since it has two distinct elements in common with W, it follows that a lies in W.

Then by Theorem 149 P and a may be taken as determining W and any element of W lies either in P or in a general plane parallel to P and intersecting a.

If now we call this set of mutually parallel general planes the "primary set" we have already seen in proving Theorem 149 that Q must either be identical with some general plane of the primary set or else must have a general line in common with each general plane of the primary set.

But now, since P and Q are supposed to be distinct, Q cannot be identical with P, and since Q is supposed to have an element in common with P, it follows that Q is not parallel to P.

Thus Q cannot be identical with any general plane of the primary set and therefore must have a general line in common with each of them, including P.

Thus P and Q must have a second element in common.

Remarks

It is further evident from the above considerations that *if two distinct general planes P and Q both lie in a general threefold W, then if P and Q have no element in common they must be parallel to one another.*

Now we have already seen that we can have a separation plane S and an inertia plane P having an element O in common and which are completely normal to one another.

We have seen that in this case P and S cannot have a second element in common.

It follows that P and S cannot lie in one general threefold.

Now let a_1 and a_2 be any two distinct general lines lying in P and passing through O.

Then S and a_1 determine a general threefold, say W_1, while S and a_2 determine a general threefold, say W_2.

Now W_1 and W_2 must be distinct, for if W_2 were identical with W_1, then W_1 would contain both a_1 and a_2 and would therefore contain P.

But W_1 contains S, and so this is impossible.

Thus W_1 and W_2 are distinct general threefolds each of which contains the separation plane S.

Since there are an infinite number of general lines lying in P and passing through O, it follows that *there are an infinite number of general threefolds which all contain any separation plane S.*

Similarly *there are an infinite number of general threefolds which all contain any inertia plane P.*

Without Post. XIX or some equivalent we cannot from our remaining postulates show that there is more than one general threefold; for the proof of the existence of an inertia plane which is completely normal to a separation plane depends upon Post. XIX.

Theorem 151

If a general plane P and a general line a both lie in a general threefold W and if a does not lie in P, then either a is parallel to a general line in P or else has one single element in common with P.

Let B be any element lying in P but not in a.

Then a and B determine a general plane, say Q, which must lie in W, since it contains three elements in common with W which do not all lie in one general line.

But since P and Q have the element B in common and both lie in W, therefore by Theorem 150 they have a general line in common which we may denote by b.

Since then b must pass through the element B which does not lie in a, it follows that a and b are two distinct general lines lying in Q and must therefore either be parallel to one another, or else have one element in common, which is also an element of P.

Thus a is either parallel to a general line in P or has an element in common with P.

Further, a cannot have more than one element in common with P, since then it would require to lie in P.

Theorem 152

If a, b and c be any three distinct general lines having an element O in common, but not all lying in one general plane, and if a general line d, also passing through O, be normal to a, b and c, then d is normal to every general line in the general threefold containing a, b and c.

Let P be the general plane containing b and c.

Then a intersects P in O and so P and a determine a general threefold, say W, containing a, b and c.

Consider now any general line e in W which passes through O but is distinct from a, b and c.

Then a and e determine a general plane, say Q, which, by Theorem 147, must lie in W.

Further, Q cannot be identical with P, since Q contains a but P does not contain it.

Again Q and P have the element O in common and therefore, by Theorem 150, they have a general line, say f, in common which passes through O.

Now, since d is normal to the two intersecting general lines b and c, it follows that d is normal to every general line in P and therefore is normal to f.

Again, since d is normal to the two intersecting general lines a and f, it follows that d is normal to every general line in Q and therefore is normal to e.

But e is any general line in W which passes through O but is distinct from a, b and c, and so d is normal to every general line in W which passes through O.

Next let e be any general line in W which does not pass through O and let e' be the general line through O parallel to e.

Then, by Theorem 148, e' must also lie in W and so by the first case d is normal to e' and therefore also normal to e.

Thus d is normal to every general line in W, as was to be proved.

Definition. A general line which is normal to every general line in a general threefold will be said to be *normal to the general threefold.*

Since, by Theorem 145, if three distinct general lines not all lying in one general plane have an element O in common there is at least one general line passing through O and normal to all three, it follows that through any element of a general threefold there is always at least one general line which is normal to the general threefold.

THE THREE TYPES OF GENERAL THREEFOLD

As in the case of general lines and general planes there are three types of each, so too there are three types of general threefold.

This may be shown in the following way:

If S be any separation plane and O be any element in it, there is an inertia plane, say P, which passes through O and is completely normal to S.

Now if a be any general line in P which passes through O, then a must be normal to S and must intersect it.

But a may be either:

(1) a separation line,

or (2) an optical line,

or (3) an inertia line,

and if a general threefold be determined by S and a, then these three cases give rise to the three different types.

Let W be the general threefold determined by a and S and consider first the case where a is a separation line.

If now e be any general line in W which passes through O and is distinct from a, then a and e determine a general plane Q which lies in W, and, since Q has the element O in common with S, it must have a general line, say f, in common with S.

Now f must pass through O and since it lies in S therefore a must be normal to f.

But a and f are both separation lines and we already know that if two intersecting separation lines are normal to one another they must lie in a separation plane.

Thus Q must be a separation plane and therefore e must be a separation line.

Thus every general line in W which passes through O must be a separation line.

If e' be any other general line in W which does not pass through O, then there is a general line through O parallel to e' which, by Theorem 148, must also lie in W and therefore must be a separation line.

But a general line parallel to a separation line must itself be a separation line and so e' is a separation line.

Thus every general line in W is a separation line and so no element of W is either *before* or *after* any other element of it.

It also follows from this that every general plane in W must be a separation plane.

Consider next the case where a is an optical line.

As before let e be any general line in W which passes through O and is distinct from a.

Then a and e determine a general plane Q which has a general line f in common with S.

As before a is normal to f, but in this case a is an optical line while f is a separation line and we know that in these circumstances a and f must lie in an optical plane.

Thus Q must be an optical plane and, since there is only one optical line in an optical plane which passes through any element of it and all other general lines in it which pass through that element are separation lines, it follows that e must be a separation line.

Again let e' be any other general line in W which does not pass through O.

Then there is a general line through O parallel to e' and this general line must either be the optical line a or a separation line.

Thus e' must be either an optical line or a separation line.

Again if O' be any element of W distinct from O, then O' may or may not lie in a.

If O' does not lie in a, then OO' is a separation line and there is an optical line through O' parallel to a which, by Theorem 148, must lie in W.

Thus there is at least one optical line passing through any element of W and lying in W.

Let e' be any general line in W which passes through O' but not through O, and which is not parallel to a.

Then the general line through O parallel to e' cannot be identical with a and therefore must be a separation line.

Thus e' must be a separation line.

It follows that of all the general lines passing through any given element of W and lying in W one and only one is an optical line and all the others are separation lines.

Further, all the optical lines in W are parallel to one another.

Since there are two optical lines in any inertia plane which pass through any element of it, it follows that no inertia plane can lie in W.

Thus every general plane in W must be either a separation plane or an optical plane.

It follows that all the optical lines in W being parallel to one another must be neutral parallels.

Consider finally the case where a is an inertia line.

As before let e be any general line in W which passes through O and is distinct from a.

Then a and e determine a general plane Q, which lies in W and, since a is an inertia line, Q must be an inertia plane.

Thus e may be either an inertia line, an optical line, or a separation line.

If O' be any element in W which is distinct from O and if d be any general line passing through O and lying in W but distinct from OO', then through O' there is a general line parallel to d, which must lie in W and must be of the same type as d.

Thus through any element of W there are general lines of all three types lying in W.

Again, if f be any general line lying in S and passing through O, then, since a is an inertia line, a and f must lie in an inertia plane, say R.

Now, since there are an infinite number of general lines such as f which lie in S and pass through O, there must be an infinite number of inertia planes such as R which are all distinct but have the inertia line a in common.

In any one of these inertia planes such as R there are two and only two optical lines which pass through O.

All these optical lines must be distinct since the inertia planes have only an inertia line in common, and so there are an infinite number of optical lines passing through O and lying in W.

Further, any optical line which passes through O and lies in W must clearly lie in one of this set of inertia planes.

Again, if O' be any element of W distinct from O and if g be any optical line passing through O and lying in W but distinct from OO', then there is an optical line through O' parallel to g and lying in W.

The general line OO' either may or may not itself be an optical line.

Thus through any element of W there are an infinite number of optical lines which lie in W.

Now we have already seen that W contains the separation plane S and also contains inertia planes, and we can easily show that it also contains optical planes.

Thus let P be any inertia plane in W and let A be any element in W but not in P.

Then through A there is an inertia plane parallel to P which we may call P'.

Let B be the representative of A in P and let c_1 and c_2 be the two generators of P which pass through B.

Then A is neither *before* nor *after* any element of either c_1 or c_2 and so A and c_1 lie in one optical plane, say T_1, while A and c_2 lie in another optical plane, say T_2.

But T_1 and T_2 each contain three elements in common with W which do not all lie in one general line and so, by Theorem 147, both T_1 and T_2 lie in W.

Thus W contains all three types of general plane.

We thus see that there are at least three types of general threefold and we have investigated a few of their characteristic properties.

We have next to show that any general threefold must belong to one of these three types.

Since any four distinct elements which do not all lie in one general plane lie in one and only one general threefold, it will be sufficient if we examine the nature of any such general threefold.

Sets of four elements which determine the different types of general threefold

Let A, B, C, D be any four distinct elements which do not all lie in one general plane.

Then no three of them can lie in one general line and A, B and C must determine a general plane which we shall call P.

Now P may be either:

(1) an inertia plane,

or (2) an optical plane,

or (3) a separation plane.

Suppose first that P is an inertia plane and that D is any element outside it.

Let W be the general threefold containing A, B, C and D and which must evidently contain P.

Then, by Theorem 142, there is one single general line passing through D and normal to P which has an element in common with P.

Let this element be denoted by O and let a be any inertia line in P which passes through O, while b is the separation line in P and passing through O which is normal to a.

Then, since a is an inertia line, the general line DO which is normal to it must be a separation line.

But DO is also normal to b and, since we know that two intersecting separation lines which are normal to one another must lie in a separation plane, it follows that DO and b lie in a separation plane which we shall call S.

Now S contains DO and b and therefore contains three elements in common with W which do not all lie in one general line.

It follows, by Theorem 147, that S lies in W.

Thus, by Theorem 149, the general threefold determined by S and a is identical with the general threefold determined by P and DO.

This latter is however identical with W and so S and a determine W.

But a is an inertia line which is normal to the two intersecting separation lines DO and b which lie in S and therefore a is normal to S.

Thus the general threefold W is of the third type.

Further, it is evident that if any general threefold contains an inertia plane it must belong to the third type.

Next consider the case where P is an optical plane and D an element outside it.

Two sub-cases arise here: we may have

D *before* or *after* some element of P,

or D neither *before* nor *after* any element of P.

We shall suppose first that D is either *before* or *after* some element of P and we shall denote the generator of P which passes through this element by a.

If, as before, W denote the general threefold containing A, B, C and D, then W will contain P and will therefore contain a.

But, since a is an optical line and D is an element which does not lie in a but is either *before* or *after* some element of a, it follows that a and D lie in an inertia plane, say Q.

But Q contains three elements in common with W which do not all lie in one general line and so Q must lie in W.

But Q is an inertia plane and so it follows that in this case also W is a general threefold of the third type.

We shall next take the case where P is an optical plane and the element D is neither *before* nor *after* any element of P.

Let b be any separation line in P and a be any optical line in P and let b and a intersect in the element O.

If, as before, W denote the general threefold containing A, B, C and D, then W will contain P and therefore will contain a and b.

Now, since D is neither *before* nor *after* any element of P, it is neither *before* nor *after* any element of b and so D and b lie in a separation plane which we may call S.

Further, since S has three elements in common with W which do not all lie in one general line, it follows that S lies in W.

Again, D and a must lie in an optical plane and, since DO is a separation line while a is an optical line, it follows that a is normal to DO.

But a must also be normal to b for a similar reason and so, since DO and b are intersecting separation lines in S, it follows that a is normal to S

But, by Theorem 149, the general threefold determined by S and a is identical with that determined by P and DO which again is identical with W.

Since however S is a separation plane while a is an optical line normal to it, it follows that W is in this case a general threefold of the second type.

Consider next the case where P is a separation plane and, as in the previous cases, let W denote the general threefold containing A, B, C and D and therefore also containing P.

Three sub-cases occur here; thus we may have:

D neither *before* nor *after* any element of P,
or D either *before* or *after* one single element of P,
or D either *before* or *after* at least two elements of P.

Now, by Theorem 142, there is one single general line passing through D and normal to P which has an element in common with P.

Let O be this element.

Then DO may be either a separation line, an optical line, or an inertia line.

Consider first the case where D is neither *before* nor *after* any element of P.

Then D is neither *before* nor *after* O and so DO is a separation line and the general threefold W is of the first type.

Next consider the case where D is either *before* or *after* one single element of P and denote this element by O'.

Let b and c be two distinct separation lines in P and passing through O'.

Then DO' and b lie in an optical plane and DO' and c lie in another optical plane.

Since D is either *before* or *after* O', it follows that DO' is an optical line and therefore is normal to both b and c.

Since b and c intersect one another, it follows that DO' is normal to P and therefore O' must be identical with O.

Thus in this case the general threefold W is of the second type.

Next let D be either *before* or *after* at least two distinct elements of P, say E and F.

Then EF is a separation line and D does not lie in it, and so the three elements D, E and F lie in an inertia plane, say Q.

But D, E and F are elements in W and therefore Q must lie in W.

Thus, since Q is an inertia plane, it follows that the general threefold W belongs in this case to the third type.

This exhausts all the possibilities which are open and so we see that any general threefold whatever must be of one of the three types which we have considered.

We shall accordingly give special names to these three types.

Definition. If a separation line a intersects a separation plane S and is normal to it, then the aggregate of all elements of S and of all separation planes parallel to S which intersect a will be called a *separation threefold.*

Definition. If an optical line a intersects a separation plane S and is normal to it, then the aggregate of all elements of S and of all separation planes parallel to S which intersect a will be called an *optical threefold.*

Definition. If an inertia line a intersects a separation plane S and is normal to it, then the aggregate of all elements of S and of all separation planes parallel to S which intersect a will be called an *inertia threefold.**

We are now in a position to introduce a new postulate which limits the number of dimensions of our set of elements.

POSTULATE XX. **If W be any optical threefold, then any element of the set must be either before or after some element of W.**

If W be any optical threefold and A be any element of W, then through A there is one single optical line which lies in W and A is *before* certain elements of this optical line and is *after* certain others.

Thus in this case A is *before* certain elements of W and *after* certain other elements of W.

If, on the other hand, A be any element outside W, then, by Post. XX, A must be either *before* some element of W or *after* some element of W.

If A be *before* the element B of W, then there is an optical line, say b, passing through B and lying in W.

If b' be the optical line through A parallel to b, then b' will be a before-parallel of b.

But any element of W which does not lie in b must lie in an optical line c neutrally parallel to b and lying in W and so, by Theorem 26, b' must be a before-parallel of c.

Thus A must be *before* certain elements of c and, since A is not an element of W and therefore not an element of c, it follows that A cannot be *after* any element of c.

* In the first edition of this work the term *rotation* threefold was used instead of *inertia* threefold. The change was made in order that the nomenclature might be more systematic.

Thus A is *before* elements of every optical line in W and is not *after* any element of W.

Similarly if A be any element outside W and *after* some element of W, then A will be *after* elements of every optical line in W and will not be *before* any element of W.

Definition. An optical line which lies in an optical or inertia threefold will be spoken of as a *generator* of the optical threefold or inertia threefold, as the case may be.

THEOREM 153

If P be an optical plane and O be any element in it, there is only one general plane passing through O and completely normal to P.

Let a be the generator of P which passes through O and let b be any separation line in P and passing through O.

Then we already know that there is at least one optical plane, say Q, which passes through O and is completely normal to P.

Further this optical plane Q contains a.

Now let c be any separation line passing through O and lying in Q.

Then c is normal to both a and b.

Let d be any other general line which passes through O and is normal to P and let X be any element in d distinct from O.

Now, P and c determine an optical threefold, since no element of c with the exception of O is either *before* or *after* any element of P.

Let this optical threefold be denoted by W.

Then, by Post. XX, the element X is either *before* or *after* some element of W.

If X were outside W, then, as we have seen, X would be *before* or *after* elements of every generator of W and therefore *before* or *after* elements of a.

Since the general line d could not then be either identical with a or be a separation line normal to a, it follows that d could not be normal to P, contrary to hypothesis.

Thus X must lie in W and therefore d must lie in W.

But now c and d determine a general plane Q' which has three elements in common with W which are not all in one general line and therefore Q' must lie in W.

Further, since P and Q' have the element O in common, therefore by Theorem 150, they have a general line in common, which we may call a'.

But now b is normal to both c and d and, since these intersect in O, it follows that b is normal to Q' and therefore normal to a'.

But b and a' lie in the optical plane P and, since b is a separation line, a' must be an optical line.

Thus, since a' passes through O, it must be identical with a and so Q' must be identical with Q.

It follows that d lies in Q and accordingly every general line which passes through O and is normal to P must lie in Q.

Thus any general plane which passes through O and is completely normal to P must be identical with Q, or there is only one general plane passing through O and completely normal to P.

Theorem 154

If P be a separation plane and O be any element in it, there is only one general plane passing through O and completely normal to P.

We already know that there is at least one inertia plane, say Q, passing through O and completely normal to P.

Suppose, if possible, that there is a general line, say a, passing through O and normal to P but not lying in Q.

Then Q and a will determine an inertia threefold, say W.

If b and c be any two distinct general lines in P which both pass through O, then b and c will each be normal to three distinct general lines passing through O and lying in W but not all lying in one general plane.

Thus, by Theorem 152, b and c must each be normal to every general line in W.

But now we have seen that any inertia threefold contains optical planes and so there would always be at least one optical plane, say R, passing through O and lying in W.

But then both b and c would be normal to every general line in R and, since b and c are intersecting general lines in P, we should have every general line in R normal to every general line in P.

Thus P would be completely normal to R and would pass through the element O in it.

But P is a separation plane and we already know by Theorem 153 that there could be only one general plane passing through O and completely normal to R, and that one must itself be an optical plane and could not be a separation plane.

Thus the assumption that there is a general line a passing through O and normal to P but not lying in Q, leads to a contradiction and therefore is not true.

It follows that every general line passing through O and normal to P must lie in Q.

Thus Q is the only general plane which passes through O and is completely normal to P.

Thus the theorem is proved.

THEOREM 155

If P be an inertia plane and O be any element in it, there is only one general plane passing through O and completely normal to P.

We already know that there is at least one separation plane, say Q, passing through O and completely normal to P.

Let b be any separation line in Q which passes through O and let c be the one separation line lying in Q and passing through O which is normal to b.

Suppose now, if possible, that there is a general line d passing through O and normal to P but not lying in Q.

Then, since any inertia line in P would be normal to d, it would follow that d must be a separation line and, since then any inertia line in P which passed through O would be conjugate to the two intersecting separation lines b and d, it would follow, as a consequence of Theorem 99, that b and d must lie in a separation plane, say Q'.

Now Q' would require to be distinct from Q, since d is supposed not to lie in Q.

Since however we should then have two intersecting separation lines in Q', namely b and d, normal to P, it would follow that Q' was completely normal to P.

Now suppose c' to be the one separation line in Q' and passing through O which would be normal to b.

Then c and c' would be distinct separation lines, since b is the only general line common to Q and Q'.

Further, since any inertia line in P which passes through O would be conjugate to both c and c', it follows that c and c' would lie in a separation plane, say S.

But now P and b would determine an inertia threefold, say W, and since both c and c' would be normal to P and to the separation line b (which does not lie in P), it follows, by Theorem 152, that both c and c' would be normal to every general line in W.

But, as we have seen, there is at least one optical plane passing through O and lying in W, and if T be such an optical plane we should have both c and c' normal to T.

Thus the separation plane S would be completely normal to T and

this we know by Theorem 153 is impossible, since only an optical plane can have an element in common with an optical plane and be completely normal to it.

It follows that no such general line as d can exist and so every general line which passes through O and is normal to P must lie in Q.

Thus Q is the only general plane which passes through O and is completely normal to P and so the theorem is proved.

Remarks

Combining these last three theorems we get the general result:

If P be any general plane and O be any element in it, there is one and only one general plane Q passing through O and completely normal to P.

Further:

If P be an optical plane, Q is an optical plane.

If P be a separation plane, Q is an inertia plane.

If P be an inertia plane, Q is a separation plane.

Again we know that if O' be any element outside P there is at least one general plane through O' which is completely normal to P.

If we call this general plane Q', then Q' is either identical with Q or parallel to Q according as O' does or does not lie in Q.

Now there cannot be any other general plane than Q' which passes through O' and is completely normal to P.

For if Q'' were such another general plane it would either pass through O or else there would be a general plane parallel to Q'' and passing through O, which would also be completely normal to P.

Thus there would be two distinct general planes passing through O and completely normal to P; which is impossible.

Thus we can say:

If P be any general plane and O be any element of the set, there is one and only one general plane passing through O and completely normal to P.

Theorem 156

(1) *If P be an inertia or separation plane and O be any element outside it, then the general plane through O and completely normal to P has one single element in common with P.*

(2) *If P be an optical plane and O be any element outside it, then the optical plane through O and completely normal to P has an optical line in common with P if O be neither* before *nor* after *any element of P and has no element in common with P if O be either* before *or* after *any element of P.*

Let P be an inertia or separation plane and O any element outside it.

Then, by Theorem 142, there is one single general line passing through O and normal to P which has an element in common with P.

Let O' be this element.

Then, by Theorem 155 or 154, there is one single separation or inertia plane, say Q, which passes through O' and is completely normal to P; and Q has only one element in common with P.

Thus Q must contain the general line $O'O$ and therefore it must be identical with the one single general plane which passes through O and is completely normal to P.

Thus the general plane through O and completely normal to P has one single element in common with P, and so the first part of the theorem is proved.

Next let P be an optical plane and O any element outside it.

Then, by Theorem 143, if O be neither *before* nor *after* any element of P there is one single generator of P such that every general line which passes through O and intersects this generator is normal to P.

Thus if a be this generator and O' be any element in a, the general lines a and OO' determine an optical plane, say Q, which passes through O, is completely normal to P and has the optical line a in common with P.

Since there is only one optical plane through O and completely normal to P, this must be identical with Q and it has the optical line a in common with P if O be neither *before* nor *after* any element of P.

Next consider the case where O is either *before* or *after* some element of P.

Here, by Theorem 143, there is no general line passing through O and having an element in common with P which is normal to P.

Thus the optical plane through O and completely normal to P can, in this case, have no element in common with P.

Thus all parts of the theorem are proved.

Theorem 157

If a general line a have an element O in common with a general threefold W, then there is at least one general plane lying in W and passing through O to which a is normal.

Let Q be any general plane in W and passing through O.

Then, by Theorem 144 there is at least one general line, say b, passing through O and lying in Q which is normal to a.

Let c be any other general line distinct from b, lying in Q and passing through O, and let A be any element lying in W but not in Q.

Then c and A determine a general plane, say R, which must lie in W, since it contains three elements in common with W which do not all lie in one general line.

Further, R must be distinct from Q, since R contains the element A which does not lie in Q, and moreover R does not contain b.

But again, by Theorem 144, there is at least one general line, say d, passing through O and lying in R which is normal to a.

Then d must be distinct from b which it intersects in the element O and so d and b determine a general plane, say P, which must lie in W, since it contains three elements in common with W which do not all lie in one general line.

But, since a is normal to the two intersecting general lines d and b, therefore a is normal to P, and thus there is at least one general plane P lying in W and passing through O to which a is normal.

It is to be observed in connexion with the above theorem that if a were normal to any other general line passing through O and lying in W but not in P, then, by Theorem 152, a would be normal to every general line in W.

It is also to be observed that the above theorem holds both when the general line a lies in W and when it has only one element in common with W.

Theorem 158

(1) *If W be a general threefold and P be a general plane lying in W, while O is any element in P, then there is at least one general line passing through O and lying in W which is normal to P.*

(2) *There is only one such general line except in the case where W is an optical threefold and P an optical plane, in which case there are an infinite number.*

To prove the first part of the theorem consider first the case where P is an optical plane.

In this case the generator of P which passes through O is normal to P and lies in W.

Next let P be an inertia or separation plane and let A be any element lying in W but not in P.

Then by Theorem 142 there is one single general line passing through A and normal to P which has an element in common with P.

Let B be this element.

Then the general line AB has two distinct elements in common with W and therefore lies in W, but does not lie in P.

If B should be identical with O, then AB passes through O, lies in W and is normal to P.

If B be not identical with O, then there is a general line passing through O and parallel to AB which must also be normal to P.

But, by Theorem 148, this general line must also lie in W.

Thus in all cases there is at least one general line passing through O and lying in W which is normal to P.

Proceeding now to the second part of the theorem, let us consider first the case where P is either an inertia or separation plane.

Suppose, if possible, that a and b are two distinct general lines both of which pass through O, lie in W and are normal to P.

Then a and b would determine a general plane, say Q, which would have three elements in common with W not all lying in one general line, and so Q would lie in W.

Thus, by Theorem 150, since Q and P have the element O in common, they would have a general line in common.

But, since Q is supposed to contain the two intersecting general lines a and b each of which is normal to P, it would follow that Q must be completely normal to P, and since P is by hypothesis either an inertia or separation plane, it would follow that Q must be either a separation or inertia plane.

But we already know that if an inertia plane and a separation plane be completely normal to one another, they cannot have more than one element in common.

Thus P and Q could not have a general line in common, and so the supposition that more than one general line can pass through O, lie in W, and be normal to P leads in this case to a contradiction and therefore is not true.

Thus if P be an inertia or separation plane there cannot be more than one such general line.

Suppose next that P is an optical plane and let a be the generator of P which passes through O and let b be any other general line lying in W but not in P and which passes through O.

Let A be any element in b distinct from O.

Then if A be either *before* or *after* any element of P the general threefold W must be an inertia threefold and a and A must lie in an inertia plane.

Thus, since a is an optical line and, since b intersects a and lies in an

inertia plane with it, it follows that b cannot be normal to a and therefore cannot be normal to P.

Further, since a is the only general line in P which passes through O and is normal to P, it follows that in this case there is only one general line in W which passes through O and is normal to P.

Consider now the case where the element A is neither *before* nor *after* any element of P.

In this case the general threefold W must be an optical threefold and the general line b must be a separation line.

Let c be any general line in P and passing through O but distinct from a.

Then c is a separation line and b and c determine a separation plane, say S, which must lie in W.

Now a must, in this case, be normal to both b and c and therefore normal to S.

Let d be the one single separation line in S which passes through O and is normal to c.

Then d is normal to both a and c and therefore is normal to P.

If then Q be the general plane containing a and d, it contains two intersecting general lines each of which is normal to P and therefore it follows that Q is completely normal to P.

Thus every general line which passes through O and lies in Q must be normal to P.

But, since a and d are two intersecting general lines which both lie in W, it follows that Q contains three distinct elements in common with W which do not all lie in one general line and therefore Q must lie in W.

Thus in this case there are an infinite number of general lines which pass through O, lie in W, and are normal to P.

This exhausts all the different cases and so the second part of the theorem is proved.

Theorem 159

If W be a general threefold and O be any element which does not lie in it, then:

(1) *If W be an inertia or separation threefold there is one single general line passing through O and normal to W which has an element in common with W.*

(2) *If W be an optical threefold there is no general line passing through O and normal to W which has an element in common with W.*

If W be an inertia threefold it contains inertia lines.

Let f be any inertia line in W.

Then f and O lie in an inertia plane, say R, and if a be any inertia line in R and passing through O but not parallel to f, then a and f will intersect in some element, say A, which is an element of W.

If on the other hand W be a separation threefold, let A be *any* element in W and let a be the general line OA.

Now whether W be an inertia or separation threefold, it follows, by Theorem 157, that there is at least one general plane, say P, lying in W and passing through A to which a is normal.

Now if W be an inertia threefold, a has been selected so as to be an inertia line and, since only separation lines can be normal to an inertia line, it follows that P is a separation plane.

If on the other hand W be a separation threefold it can contain no other type of general plane, and so in this case also P must be a separation plane.

Now, by Theorem 158, whether W be an inertia or a separation threefold, there is one general line, say b, passing through A and lying in W which is normal to P, and, since P is a separation plane, b must intersect it.

Now a and b must be distinct, since b lies in W while a can only have the one element A in common with W.

Thus a and b lie in a general plane, say Q, and, since Q contains two intersecting general lines each of which is normal to P, it follows that Q must be completely normal to P.

Further, since P is a separation plane, it follows that Q is an inertia plane.

Now, since b is normal to P and lies in W, the general threefold W might be determined by P and b and we know that if b be a separation line, W must be a separation threefold, while if b be an optical line, W must be an optical threefold, and if b be an inertia line, W must be an inertia threefold.

It follows that if W be an inertia threefold then b must be an inertia line, while if W be a separation threefold, b must be a separation line.

But now in either of these cases there is a general line, say c, which passes through O, lies in Q and is normal to b, and in both cases c intersects b in some element, say O', which is an element of W.

Further, c will be a separation line if b be an inertia line: that is, if W be an inertia threefold; while c will be an inertia line if b be a separation line: that is, if W be a separation threefold.

Now, since c lies in Q, and since Q is completely normal to P, it follows that c is normal to P.

If then P' be a general plane passing through O' and parallel to P or identical with it, it follows, by Theorem 148, that P' must also lie in W.

Thus c will be normal to P' and to the general line b which intersects P' in O'.

It is thus evident that c is normal to three distinct general lines in W which have the element O' in common and which do not all lie in one general plane and therefore, by Theorem 152, c is normal to W.

Also c passes through O and has the element O' in common with W.

Now there can be no other general line passing through O and normal to W; for suppose, if possible, that c' is such another general line.

Then c and c' would determine a general plane, say T, which would contain two intersecting general lines each of which would be normal to every general line in W and therefore normal to every general plane in W.

Thus T would be completely normal to every general plane in W.

But through any element of W there passes more than one general plane which lies in W and so we should have more than one general plane passing through any element of W and completely normal to T, which, as we have seen, is impossible.

Thus the supposition that more than one general line can pass through O and be normal to W leads to a contradiction and therefore is not true.

Thus there is one and only one general line which passes through O and is normal to W when W is an inertia or separation threefold, and this general line has an element in common with W.

Suppose next that W is an optical threefold.

Then, by Post. XX, O must be either *before* or *after* some element of W and, as we have seen, if O be *before* any element of W it must be *before* elements of every generator of W, while if O be *after* any element of W it must be *after* elements of every generator of W.

If then a be any general line which passes through O and has an element A in common with W, then A must lie in some generator of W, say f, and f and a will lie in an inertia plane.

But, since f is an optical line and a is a general line intersecting f and lying in an inertia plane with it, it follows that a cannot be normal to f and therefore cannot be normal to W.

Thus in this case there is no general line passing through O and normal to W which has an element in common with W.

Thus both parts of the theorem are proved.

Remarks

If W be an inertia or separation threefold and O be any element *in* W, it is easy to see that there is one and only one general line passing through O and normal to W.

For if A be any element outside W and a be the one general line passing through A and normal to W, then a will have an element B in common with W.

If B should coincide with O, then a is a general line passing through O and normal to W.

If B does not coincide with O, then a general line a' passing through O and parallel to a must be normal to every general line in W and must therefore be normal to W.

Thus we have shown that there is at least one general line passing through O and normal to W, and the same considerations employed in the last theorem show that there is only one such general line.

Further, the general line through O normal to W cannot have more than the one element O in common with W; for if it had a second element in common with W it would lie entirely in W, and, by Theorem 148, it would follow that a must lie in W, contrary to the hypothesis that the element A of a lies outside W.

In this respect an optical threefold is quite different.

Through any element O in an optical threefold W there passes one single generator of W, say a.

Now a is normal to any separation line in W and is also normal to itself.

Thus a is normal to W and passes through O, but lies entirely in W.

If O' be any element outside W and a' be an optical line parallel to a, then a' is also normal to W but can have no element in common with W.

We may also show, by similar considerations to those employed in the case of an inertia or separation threefold, that there cannot be more than one general line passing through any element and normal to a given optical threefold.

Thus for all three types of general threefold we have the result:

If W be any general threefold and O be any element of the set, there is one and only one general line passing through O and normal to W.

THEOREM 160

If a be a general line and O be any element in it, there is one and only one general threefold passing through O and normal to a.

Let P be any inertia plane containing a and let Q be the separation plane passing through O and completely normal to P.

Then P and Q have only the one element O in common.

Now through O and lying in P there is one single general line, say b, which is normal to a.

But b and Q can have only one element in common and therefore they determine a general threefold, say W.

Since, however, a is normal to every general line in Q and is also normal to the general line b which passes through O and does not lie in Q, it follows, by Theorem 152, that a is normal to W.

Thus there is at least one general threefold passing through O and normal to a.

We shall next show that every general line which passes through O and is normal to a must lie in W.

Since every such general line which lies in Q must lie in W, it will be sufficient to consider any general line c passing through O normal to a and not lying in Q.

Then c and Q determine a general threefold, say W', and by Theorem 158 there is at least one general line, say d, passing through O and lying in W' which is normal to Q.

Further, since Q is a separation plane, d must lie in the inertia plane through O which is completely normal to Q, and since there is only one such inertia plane, it follows that d must lie in P.

But, since a is normal to c and Q, it follows that a is normal to W' and therefore is normal to d.

But there is only one general line passing through O and lying in P which is normal to a, and by hypothesis b is this general line.

It follows that d must be identical with b and so, by Theorem 149, since d and Q must determine the same general threefold as do c and Q, it follows that W' must be identical with W.

Thus c must lie in W.

But if there were any other general threefold distinct from W which passed through O and was normal to a, such general threefold would require to contain a general line which passed through O and was normal to a but which did not lie in W, and this we have shown to be impossible.

Thus there is one and only one general threefold which passes through O and is normal to a.

Remarks

In the above theorem it is to be observed that: if a be an inertia line, b must be a separation line; if a be a separation line, b must be an inertia line; while if a be an optical line, b must be the same optical line.

Thus it follows that: if a be a general line and O be any element in it, while W is a general threefold passing through O and normal to a, then:

(1) If a be an inertia line, W is a separation threefold.
(2) If a be a separation line, W is an inertia threefold.
(3) If a be an optical line, W is an optical threefold containing a.

On the other hand we have already seen that if W be a general threefold and O be any element in it, there is one and only one general line a passing through O and normal to W.

Thus it follows that:

(1) If W be a separation threefold, a is an inertia line.
(2) If W be an inertia threefold, a is a separation line.
(3) If W be an optical threefold, a is an optical line lying in W.

Again if a be a general line and O be any element which does not lie in a, then, through O there is one single general line, say a', which is parallel to a and is accordingly a general line of the same type.

Thus through O there is a general threefold which is normal to a' and therefore also normal to a.

Further, there cannot be a second general threefold passing through O and normal to a, for such general threefold would also be normal to a' and so we should have two general threefolds passing through O and normal to a' contrary to Theorem 160.

Thus we can extend Theorem 160 and say:

If a be a general line and O be any element of the set, there is one and only one general threefold passing through O and normal to a.

Theorem 161

If W be an optical threefold and A be any element outside it, then every optical line through A, except the one parallel to the generators of W, has one single element *in common with W.*

Let a be the optical line through A parallel to the generators of W and let b be any such generator.

Then by Post. XX A must be either *before* or *after* some element of W and we have already seen that if A be *before* an element of W it must be *before* elements of every generator of W; while if A be *after* an element of W it must be *after* elements of every generator of W.

Thus a must be either a before- or after-parallel of b.

It will be sufficient to consider the case where a is a before-parallel of b since the proof in the other case is quite analogous.

Then a and b lie in an inertia plane and so there is one single optical line passing through A and intersecting b in some element, say B.

If we call this optical line c, then c has the element B in common with W.

If then d be any optical line passing through A but distinct from c and a, it follows, by Post. XII, that there is one single element in d, say D, which is neither *before* nor *after* any element of b.

Now if D were outside W it would be either *before* or *after* elements of every generator of W, as we have already seen.

Thus, since D is neither *before* nor *after* any element of the generator b, it follows that D must lie in W.

It follows that every optical line through A with the exception of a has at least one element in common with W.

But if any optical line has more than one element in common with W it must lie entirely in W, which is not possible for any optical line which passes through the element A.

It follows that every optical line through A with the exception of a has one single element in common with W, as was to be proved.

Theorem 162

If W be a general threefold and A be any element outside it, then any general line through A is either parallel to a general line in W or else has one single element *in common with W.*

It will be observed that the last theorem is a special case of this one.

Let a be any general line which passes through A.

Now a cannot have more than one element in common with W, for then it would require to lie entirely in W and therefore could not pass through A.

Let B be a second element in a distinct from A.

In case W be an inertia or separation threefold, let the general line through A normal to W meet W in the element A', as we have seen in Theorem 159 that it must.

Now in case the general line a should coincide with AA' it would have an element in common with W, and so we shall suppose it is distinct from it.

Again let the general line through B normal to W meet W in the element B'.

Then since B does not lie in AA' we must have BB' parallel to AA'.

In case W be an optical threefold, then by Theorem 161 any optical line through A except the one parallel to the generators of W must have an element in common with W.

Let any optical line through A which is not parallel to the generators of W meet W in the element A'.

In case the general line a should coincide with AA' it would have an element in common with W, and so we shall suppose it is distinct from it.

Let the optical line through B parallel to AA' meet W in the element B'.

Now both in the cases where W is an inertia or separation threefold and where W is an optical threefold, since BB' is parallel to AA', it follows that BB' and AA' lie in a general plane which we may call Q.

But $A'B'$ and a must also lie in Q, and therefore a is either parallel to $A'B'$ or intersects $A'B'$ in some element, say C.

But $A'B'$ has two distinct elements A' and B' in common with W, and therefore $A'B'$ must lie in W, and if the element C exists it must lie in W.

Thus the general line a is either parallel to a general line in W or else a has one single element in common with W.

Definition. If a general line and a general threefold have *one single element* in common, they will be said to *intersect* in that element.

Remarks

Since a separation threefold contains neither an inertia nor an optical line it is evident that it can contain no general line which is parallel to either of these.

Thus it follows from the last theorem that: *every inertia and every optical line intersects every separation threefold.*

Again, an optical threefold does not contain any inertia line, and all the optical lines which it contains are parallel to one another.

Thus: *every inertia line and every optical line which is not parallel to a generator of an optical threefold intersects the optical threefold.*

Analogous results to these may be deduced from Theorem 151, with

regard to the intersection of certain types of general lines with certain types of general planes.

Thus, since a separation plane contains neither an inertia nor an optical line, it follows from Theorem 151 that: *if W be an inertia threefold, every inertia and every optical line in W intersects every separation plane in W.*

Similarly: *if W be an inertia threefold, every inertia line in W and every optical line in W which is not parallel to a generator of an optical plane in W intersects the optical plane.*

Again: *if W be an optical threefold, every optical line in W intersects every separation plane in W.*

Theorem 163

If W be a general threefold and P be a general plane which does not lie in W, then if P has one element in common with W, it has a general line in common with W.

Let P and W have the element A in common and let B be any element in P which does not lie in W.

Let b be any general line in P which passes through B but is distinct from BA.

Then by Theorem 162 b must either intersect W in some element, say C, or else b must be parallel to some general line, say b', which lies in W.

In the first case P and W have the two distinct elements A and C in common and therefore have the general line AC in common.

In the second case a general line b'' passing through A and parallel to b' or identical with it must lie in W.

But b'' must be parallel to b and since it passes through the element A of P it must lie in P.

Thus in this case P and W have the general line b'' in common and so the theorem holds in general.

Theorem 164

If W_1 and W_2 be two distinct general threefolds having an element A in common, then they have a general plane in common.

Let B be any element which lies in W_1 but not in W_2.

Then the general line AB lies in W_1.

Let Q and R be any two distinct general planes which contain the general line AB and which lie in W_1.

Then Q does not lie in W_2 but has the element A in common with W_2 and therefore, by Theorem 163, Q has a general line, say a, in common with W_2.

Similarly R has a general line, say b, in common with W_2.

Now both a and b must be distinct from the general line AB since the latter does not lie in W_2 and, since Q and R have only the general line AB in common, it follows that b is distinct from a.

Thus a and b are two general lines intersecting in A and each of them lying both in W_1 and W_2 and so they determine a general plane, say P.

But P contains three elements in common both with W_1 and W_2 and which do not all lie in one general line and so P lies both in W_1 and W_2.

Thus W_1 and W_2 have a general plane in common.

Theorem 165

If P_1 and P_2 be two general planes having no element in common, then through any element of either of them there is at least one general line lying in that general plane which is parallel to a general line in the other general plane.

Let O_1 be any element in P_1 and let O_2 be any element in P_2 and let the general line O_1O_2 be denoted by a.

Then P_1 and a determine a general threefold, say W_1, while P_2 and a determine a general threefold, say W_2.

If W_2 should be identical with W_1, then P_1 and P_2 lie in one general threefold and, since they have no element in common, it follows, by Theorem 151, that any general line in P_1 is parallel to a general line in P_2, and so P_1 and P_2 are parallel to one another.

If W_2 be not identical with W_1, then, since W_1 and W_2 have all the elements of a in common, it follows, by Theorem 164, that they have a general plane, say Q, in common which must contain a.

But now Q must be distinct from both P_1 and P_2, for otherwise P_1 or P_2 would contain a and so P_1 and P_2 would have an element in common, contrary to hypothesis.

But now P_1 and Q both lie in W_1 and they have the element O_1 in common, and therefore, by Theorem 150, they have a general line, say b_1, in common, which passes through O_1.

Similarly P_2 and Q have a general line, say b_2, in common, which passes through O_2.

But, since b_1 and b_2 lie in P_1 and P_2 respectively, they can have no element in common and, since they both lie in the general plane Q, they must be parallel to one another.

Thus the theorem is proved.

Remarks

It is easy to see that if two general planes P_1 and P_2 have one single element O in common, then no general line in P_1 can be parallel to any general line in P_2.

For let a_1 and a_2 be two general lines in P_1 and P_2 respectively, then a_1 cannot be parallel to a_2 if both pass through O.

Further, they cannot be parallel if one passes through O and the other does not, for then they could not lie in one general plane.

Finally they cannot be parallel if neither of them passes through O, for then a general line a_1' passing through O and parallel to a_1 would lie in P_1 and so could not be parallel to a_2 as it would require to be if a_2 were parallel to a_1.

Theorem 166

If W be a general threefold and O be any element outside it, and, if further, a and b be two distinct general lines intersecting in O and each of them parallel to a general line in W, then:

(1) *The general plane containing a and b has no element in common with W.*

(2) *The general plane containing a and b is parallel to a general plane in W.*

Neither a nor b can have any element in common with W, since, by Theorem 148, if either of them had an element in common with W, it would require to lie entirely in W and so could not contain the element O.

But, if P be the general plane containing a and b, any element in P must lie either in a or in a general line parallel to a and intersecting b.

But every general line of this character must be parallel to the general line in W to which a is parallel, and therefore can have no element in common with W.

Thus P can have no element in common with W.

In order to prove the second part of the theorem, let a' and b' be general lines in W to which a and b are respectively parallel.

Then a' and b' either intersect, in which case they lie in a general

plane which lies in W and is parallel to P, or else a general line, say b'', parallel to b', may be taken through any element of a' and then b'' must lie in W, by Theorem 148.

Thus in this case a' and b'' will lie in a general plane which will lie in W and be parallel to P.

Thus in all cases P will be parallel to a general plane in W.

Definition. If W be a general threefold and if through any element A outside W a general line a be taken parallel to any general line in W, then the general line a will be said to be *parallel* to the general threefold W.

Definition. If W be a general threefold and if through any element A outside W a general plane P be taken parallel to any general plane in W, then the general plane P will be said to be *parallel* to the general threefold W.

Theorem 167

If W be a general threefold and O be any element outside it, and if through O there pass three general lines a, b, and c, which do not all lie in one general plane and which are respectively parallel to three general lines in W, then a, b and c determine a general threefold W', such that every general line in W' is parallel to a general line in W.

Let P be the general plane containing b and c.

Then, since a, b and c do not lie in one general plane, it follows that a can only have the one element O in common with P.

Now the general line a can have no element in common with W, for then, since it is parallel to a general line in W, it would, by Theorem 148, require to lie in W and so could not contain the element O.

Again, by Theorem 166, the general plane P can contain no element in common with W, nor can any general plane which is parallel to P and which intersects a.

But now any element in W' must either lie in P or in a general plane parallel to P and intersecting a.

Thus no element in W' can lie in W, and so no general line in W' can have an element in common with W.

Thus, by Theorem 162, any such general line must be parallel to a general line in W.

Similarly any general line in W must be parallel to a general line in W'.

Definition. If W be a general threefold and if through any element A outside W three general lines be taken not all lying in one general

plane but respectively parallel to three general lines in W, then the three general lines through A determine a general threefold which will be said to be *parallel* to W.

Remarks

Since a general line can only be parallel to a general line of the same kind, and since if one general threefold be parallel to another, any general line in either of them is parallel to a general line in the other, it follows that a general threefold can only be parallel to a general threefold of the same kind.

Again, if W be a general threefold and A be any element outside it, while W' is a general threefold through A parallel to W, then since W' contains the general line through A parallel to any general line in W, the general threefold W' must be uniquely determined when we know W and A.

Also, since two distinct general lines which are parallel to a third general line are parallel to one another, it follows that: *two distinct general threefolds which are parallel to a third general threefold are parallel to one another.*

Again, from Theorem 162, it is evident that: *if W be a general threefold and A be any element outside it, then any general line through A must either lie in the general threefold passing through A and parallel to W, or else must intersect W.*

If W and W' be two distinct general threefolds and if A be any element in W' but not in W, then, if W' be not parallel to W, there must be at least one general line passing through A and lying in W' which is not parallel to any general line in W and which therefore, by Theorem 162, must intersect W.

Thus W' will have an element in common with W and so, by Theorem 164, W and W' must have a general plane in common. Thus *any two distinct general threefolds must either be parallel or else must have a general plane in common.*

It is also to be noted that if a general threefold W be normal to a general line a, then any general threefold W' parallel to W must also be normal to a.

Other cases of normality

We have already considered the normality of a general line to a general line, a general plane, or a general threefold.

We have also considered the complete normality of a general plane to a general plane.

These are the only cases in our geometry in which the normality of n-folds is complete.

Thus it is not possible to have every general line in a general plane P normal to every general line in a general threefold W, for then we should have more than one general plane passing through any element of W and completely normal to P, which, as we have seen, is impossible.

For a similar reason we cannot have every general line in a general threefold W_1 normal to every general line in a general threefold W_2.

The most possible in these directions is to have a general plane P through any element of which there is one single general line lying in P which is normal to a general threefold W; or to have a general threefold W_1 through any element of which there is one single general line lying in W_1 which is normal to a general threefold W_2.

Again, we may have a general plane P_1 through any element of which there is one single general line lying in P_1 which is normal to a general plane P_2.

In these cases we have what may be described as partial normality.

In ordinary three dimensional geometry the normality of two planes is of this partial character.

Since it is desirable, so far as is possible, to have our nomenclature in conformity with that employed in ordinary geometry, we shall find it convenient to describe the general planes and general threefolds in the above cases as *normal* to one another.

Thus we may have general planes *normal* to one another or *completely normal* to one another: the expression 'normal' by itself being taken to mean partially normal.

In the case of a general plane or a general threefold which is partially normal to a general threefold the word *normal* may be used by itself without any ambiguity.

Thus we have the following definitions:

Definition. A general plane P_1 will be said to be *normal* to a general plane P_2 if through any element of P_1 there is one single general line lying in P_1 which is normal to P_2.

Definition. A general plane P will be said to be *normal* to a general threefold W if through any element of P there is one single general line lying in P which is normal to W.

Definition. A general threefold W_1 will be said to be *normal* to a general threefold W_2 if through any element of W_1 there is one single general line lying in W_1 which is normal to W_2.

It is evident in the above three definitions we might substitute the word *every* for the word *any*.

It is easy to see that if a general plane P_1 be normal to a general plane P_2, then P_2 will be normal to P_1.

It will be sufficient to consider the case where P_1 and P_2 have an element A in common.

Let a be the one single general line lying in P_1 and passing through A which is normal to P_2 and let b be any other general line in P_1 which passes through A.

Then by Theorem 144 there is at least one general line, say c, passing through A and lying in P_2 which is normal to b.

But c must also be normal to a and therefore c must be normal to P_1.

Further, there cannot be more than one general line passing through A and lying in P_2 which is normal to b, unless P_1 be completely normal and not merely normal to P_2.

Again, a separation line a may be normal to all three types of general plane and also may lie in all three types of general plane.

If then a be normal to any general plane P_1 and if P_2 be any general plane containing a but not completely normal to P_1, then P_2 will be normal to P_1.

Thus any type of general plane may be normal to any type of general plane.

In particular, since an optical plane contains a series of optical lines which are normal to it, it follows that an optical plane is normal to itself.

It is evident from the definitions that, if a general plane P be normal to a general threefold W, then P will be either simply normal or completely normal to any general plane in W.

THEOREM 168

If a general plane P be normal to a general threefold W, then through any element of W there is one single general plane lying in W and completely normal to P.

By definition, since P is normal to W, it follows that there is a general line passing through any element of P and lying in P which is normal to W.

Let O be any element of W and let a be the one single general line passing through O which is normal to W.

Now all general lines which are normal to W are co-directional, so that a is co-directional with a set of general lines in P.

Let b be any general line other than a which passes through O and is co-directional with some general line in P.

Then a and b lie in a general plane, say P', which is either parallel to P or identical with it.

But now, by Theorem 157, there is a general plane, say Q, passing through O and lying in W to which b is normal.

Since, however, a is normal to W, it follows that a is also normal to Q.

Thus we have the two intersecting general lines a and b both lying in P' and both normal to every general line in Q.

It follows that Q is completely normal to P' and, since P' is either parallel to P or identical with it, it follows that Q is completely normal to P.

Thus, since Q is taken through any arbitrary element of W and lies in W, the theorem is proved.

THEOREM 169

If a general threefold W_1 contain a general line which is normal to a general threefold W_2, then W_2 contains a general line which is normal to W_1.

Let a be a general line lying in W_1 and which is normal to W_2.

Let P_1 and P_2 be any two distinct general planes both of which contain a and lie in W_1.

Then P_1 and P_2 are both normal to W_2 and accordingly, by Theorem 168, if O be any element in W_2, there is a general plane, say Q_1, passing through O and lying in W_2 which is completely normal to P_1 and similarly there is a general plane, say Q_2, passing through O and lying in W_2 which is completely normal to P_2.

Now Q_1 and Q_2 cannot be identical, for then we should have the two distinct general planes P_1 and P_2 both containing a and both completely normal to the same general plane, which we know to be impossible.

Thus, since Q_1 and Q_2 both lie in W_2 and have the element O in common, it follows, by Theorem 150, that Q_1 and Q_2 have a general line in common which we shall call b.

Then b must be normal to both P_1 and P_2 and, since these are distinct intersecting general planes in W_1, it follows that b is normal to W_1 and lies in W_2.

The above theorem might also be stated in the form:

If a general threefold W_1 be normal to a general threefold W_2, then W_2 is normal to W_1.

Some analogies

Before proceeding with the next part of our subject we shall point out a few analogies which exist between an inertia plane, an inertia threefold and the whole set of elements.

We have seen that: if P be an inertia plane and A be any element in it there are two and only two optical lines passing through A and lying in P.

We have an analogue to this in the case of an inertia threefold.

We shall show that if W be an inertia threefold and a be any separation line in it there are two and only two optical planes containing a and lying in W.

In order to prove this: let O be any element in a.

Then, by Theorem 157, there is at least one general plane lying in W and passing through O to which a is normal.

Further, there cannot be more than one such general plane, for otherwise a would require to be normal to the inertia threefold W and would therefore intersect W contrary to the hypothesis that a lies in W.

Let P be this one general plane.

Then P cannot be a separation plane, for, since a is a separation line, this would require W to be a separation threefold, contrary to hypothesis.

Again, P cannot be an optical plane for this would require W to be an optical threefold, contrary to hypothesis.

Thus P must be an inertia plane and so there are two and only two optical lines, say c_1 and c_2, which pass through O and lie in P.

Thus a must be normal to both c_1 and c_2 and it cannot be normal to any other optical line passing through O and lying in W; for such an optical line could not lie in P, and if a were normal to such an optical line in addition to c_1 and c_2, it would require to be normal to W, which we know to be impossible.

But now c_1 and a lie in one optical plane, say R_1, while c_2 and a lie in another optical plane, say R_2.

Now R_1 and R_2 are the only optical planes in W which contain a; for the existence of a third would require the existence of a third optical line passing through O, lying in W and normal to a, which, as we have seen, is impossible.

This proves the required result.

Again we have a corresponding result for the whole set of elements.

We shall show that if S be any separation plane there are two and only two optical threefolds containing S.

For let O be any element in S and let P be the one single inertia plane which passes through O and is completely normal to S.

Further let c_1 and c_2 be the two generators of P which pass through O.

Then c_1 and c_2 are each normal to S, and accordingly c_1 and S determine one optical threefold, say W_1, while c_2 and S determine another optical threefold, say W_2.

Now W_1 and W_2 are the only optical threefolds which contain S, for the existence of a third would require the existence of a third optical line passing through O and normal to S.

But if there were three optical lines passing through O and normal to S, there would be more than one inertia plane passing through O and completely normal to S, which we have seen is impossible.

Thus there are two and only two optical threefolds containing S, and so we see that we have here a certain analogy between an inertia plane, an inertia threefold, and the whole set of elements.

It was pointed out in another part of this work, that if W be an inertia threefold and A be any element in it, then there are an infinite number of optical lines which pass through A and lie in W.

It is easy to show that if a be any separation line, there are an infinite number of optical planes which contain a, although, as we have seen, there are only two in any one inertia threefold containing a.

Thus let O be any element in a and let W be the one single inertia threefold which passes through O and is normal to a.

Then there are an infinite number of optical lines passing through O and lying in W, and each of these must be normal to a.

Thus each of these optical lines along with a determines an optical plane and all these latter must be distinct.

It follows that there are an infinite number of optical planes containing any separation line.

It is easy to show that if W be an inertia threefold and a be any optical line in it, then there is one and only one optical plane containing a and lying in W.

For let O be any element in a; then, since there are an infinite number of optical lines passing through O and lying in W, there are an infinite number of inertia planes lying in W and containing a.

Let P be any such inertia plane.

Then, by Theorem 158, there is one and only one general line, say b, passing through O and lying in W which is normal to P.

Then b must be normal to a, and so a and b determine an optical plane, say R, which lies in W.

Now R is the only optical plane which contains a and lies in W; for let R' be any other optical plane containing a.

Then any element X lying in R' but not in a would be neither *before* nor *after* any element of R, and so X and R would lie in an optical threefold and could not lie in W.

This proves that R is the only optical plane containing a and lying in W.

If now we consider the whole set of elements we can easily show that if a be an optical line there is one and only one optical threefold containing a.

In order to prove this we have only to remember that a is normal to any optical threefold containing it and, by Theorem 160, if O be any element in a, there is one and only one optical threefold passing through O and normal to a.

Again, if P be an optical plane, there is one and only one optical threefold containing P; for if A be any element which is neither *before* nor *after* any element of P, then P and A determine an optical threefold, say W.

Also W is the only optical threefold containing P, for otherwise we should have more than one optical threefold containing any optical line in P.

Theorem 170

If A, B, C, D be the corners of an optical parallelogram (AC being the inertia diagonal line) and if A, B', C, D' be the corners of a second optical parallelogram, while A', B', C', D' are the corners of a third optical parallelogram whose diagonal line $A'C'$ is conjugate to BD, then A', B, C', D will be the corners of a fourth optical parallelogram.

In order to prove this important theorem, we shall first prove the following lemma.

If O, C and C' be three distinct elements in an inertia plane P such that OC and OC' are inertia lines while CC' is a separation line, and if further, CC'' be another separation line intersecting OC' in C'', and if M be the mean of C and C' while N is the mean of C and C'', then if MO be conjugate to CC' we cannot have NO conjugate to CC''.

It will be sufficient to consider the case where O is *before* C, since the case where O is *after* C is quite analogous.

Since CC' is a separation line, while OC' is an inertia line, and since O is *before* C, it follows that O must also be *before* C'.

Let E, C, F, C' be the corners of an optical parallelogram in the inertia plane P and let F be *after* E.

Then FE is conjugate to CC' and intersects it in M and must therefore by hypothesis be identical with MO.

Now E must be *after* O, for in the first place E cannot be identical with O since EC' is an optical line while OC' is an inertia line.

Again, O cannot be *after* E, for then we should have O *after* one element of the optical line EC' and *before* another element of it without lying in the optical line, contrary to Theorem 12.

Thus since OE is an inertia line we must have E *after* O.

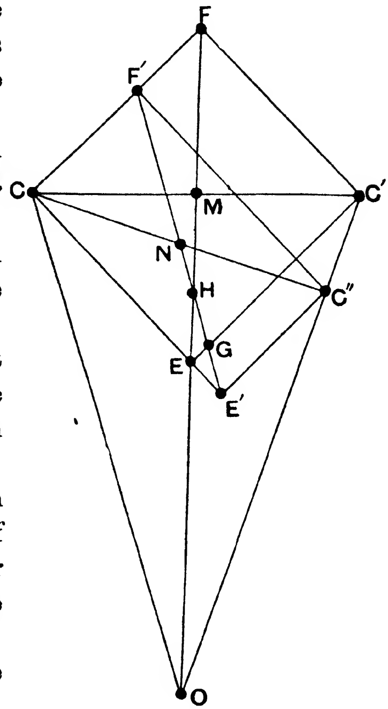


Fig. 41.

Now the element C'' is distinct from C', and since C'' and C' lie in an inertia line we must have one *after* the other.

Suppose first that C' is *after* C''.

Let the optical line through C'' parallel to $C'E$ intersect CE in E' and let the optical line through C'' parallel to $E'C$ intersect CF in F'.

Then E', C, F', C'' are the corners of an optical parallelogram, and, since CC'' is a separation line, $E'F'$ must be an inertia line conjugate to it and intersecting it in the element N.

But now $C''E'$ is a before-parallel of $C'E$ while $C''F'$ is a before-parallel of $C'F$.

Thus we must have E' *before* E and F' *before* F.

Now let the inertia line $E'F'$ intersect the optical line EC' in the element G.

Then E' is *before* E and is therefore in the β sub-set of E and so G must be in the α sub-set of E.

Thus G must be *after* E.

But, since we have also F *after* F', it follows that EF and GF' intersect in an element, say H, which is between EC' and CF.

Thus H is linearly between E and F and is therefore *after* E.

But E is *after* O and therefore H is *after* O.

Thus the conjugate to CC'' through N in the inertia plane P intersects MO in an element which is *after* O and so NO cannot be conjugate to CC''.

This proves the lemma provided that C' is *after* C''.

Next consider the case where C'' is *after* C'.

Suppose, if possible, that NO is conjugate to CC''.

Then, by the case already proved, MO could not be conjugate to CC', contrary to hypothesis, and so the lemma is proved in general.

We shall now make use of this lemma in order to prove the theorem.

We shall suppose that C is *after* A and C' *after* A'.

Now, since the first and second optical parallelograms have the pair of opposite corners A and C in common, it follows, by Theorem 60, that they have a common centre, say O.

Further, since the second and third optical parallelograms have the pair of opposite corners B' and D' in common, they have also the same centre O.

Thus AC and $A'C'$ intersect in the element O, and, since they are both inertia lines, they must lie in one inertia plane, say P.

But C and C' are distinct elements lying in the α sub-sets of the distinct elements B' and D', of which the one is neither *before* nor *after* the other, and therefore C' is neither *before* nor *after* C, and, in an analogous way, A' is neither *before* nor *after* A.

Thus CC' and AA' are both separation lines.

Let M be the mean of C and C'.

Then B', C and C' are three corners of an optical parallelogram having M as centre, while D', C and C' are three corners of another optical parallelogram of which M is the centre.

Further, MB' and MD' must both be inertia lines and are each conjugate to CC'.

Thus, by Theorem 103, CC' is conjugate to every inertia line which passes through M and lies in the inertia plane containing MB' and MD'.

But O is linearly between B' and D' while M is *after* both B' and D' but is not in the general line $B'D'$ and so, by Theorem 73(b), MO is an inertia line.

Thus, since MO is in the inertia plane containing MB' and MD', it follows that CC' is conjugate to MO.

If now we consider the optical parallelogram having B and D as opposite corners and lying in the inertia plane containing BD and $A'C'$, it follows, since O is the mean of B and D, that O must be the centre of this optical parallelogram.

Further, since by hypothesis $A'C'$ is conjugate to BD, it follows that the remaining two corners of this optical parallelogram must lie in $A'C'$.

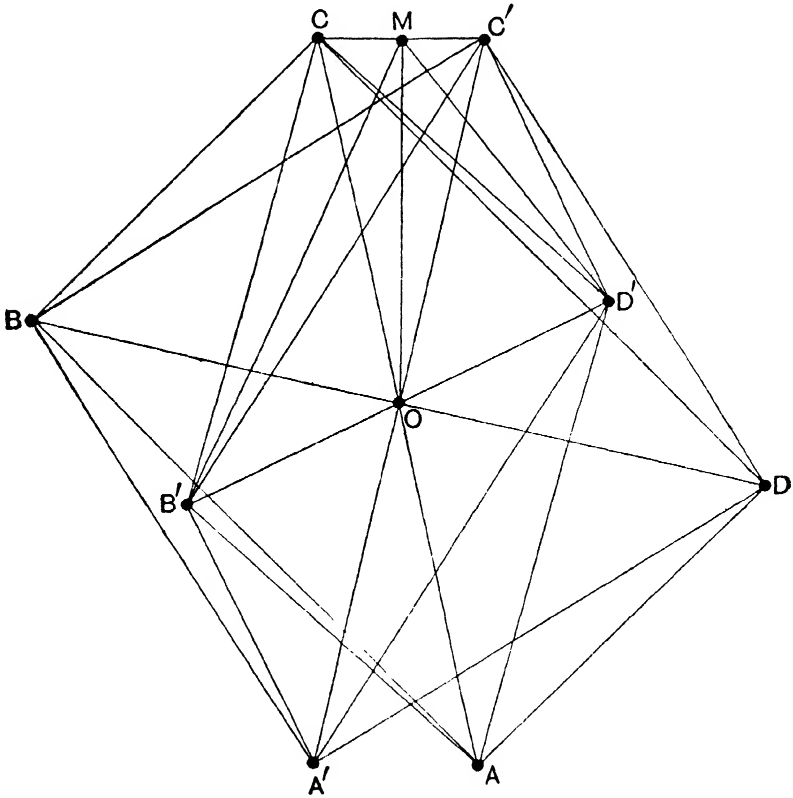


Fig. 42.

Let A'' and C'' be these remaining corners and let C'' be *after* A''.

Then just as CC' was shown to be a separation line, we may show that CC'' is a separation line, and if N be the mean of C and C'' we may show that CC'' is conjugate to NO, which may be proved to be an inertia line as was MO.

But if CC'' were distinct from CC', our lemma shows that this would not be possible, and so CC'' must be identical with CC'.

Thus, since C'' lies in $A'C'$, it follows that C'' is identical with C'.

Similarly A'' is identical with A' and therefore A', B, C', D are the corners of an optical parallelogram as was to be proved.

Theory of congruence

We are now in a position to consider the problems of *congruence* and *measurement* in our system of geometry.

The first point to be examined is the *congruence* of pairs of elements, and we shall find that there are several cases which have to be considered separately.

Two distinct elements A and B will be spoken of briefly as *a pair* and will be denoted by the symbols (A, B) or (B, A).

The order in which the letters are written will be taken advantage of in order to symbolize a certain correspondence between the elements of pairs, as we shall shortly explain.

Since any two distinct elements determine a general line, there will always be one general line associated with any given pair, but different pairs will be associated with the same general line.

If we set up a correspondence between the elements of a pair (A, B) and a pair (C, D) we might either take C to correspond to A and D to B, or else take D to correspond to A and C to B.

The first of these might be symbolized briefly by:

(A, B) corresponds to (C, D),

or (B, A) corresponds to (D, C).

The second might be symbolized by:

(A, B) corresponds to (D, C),

or (B, A) corresponds to (C, D).

If we consider the case of pairs which have a common element, say (A, B) and (A, C), and if

(A, B) corresponds to (A, C),

then the element A corresponds to itself.

Now the *congruence* of pairs is a correspondence which can be set up in a certain way between certain pairs lying in general lines of the same type.

In dealing with this subject it will be found convenient to have a systematic notation for optical parallelograms, so that we may be able to distinguish how the different corners are related.

If A, B, C, D be the corners of an optical parallelogram we shall use the notation $A\overline{BC}D$ when we wish to signify that the corners A and D lie in the inertia diagonal line and that A is *before* D, while B and C

lie in the separation diagonal line so that the one is neither *before* nor *after* the other.

If O be the centre of the optical parallelogram $A\overline{BC}D$, then it is obvious that O will be *after* A and *before* D.

Definition. A pair (A, B) will be spoken of as an *optical pair*, an *inertia pair*, or a *separation pair* according as AB is an optical, an inertia or a separation line.

We shall first give a definition of the congruence of inertia pairs having a self-corresponding element.

Definition. If $A_1\overline{BC}D_1$ and $A_2\overline{BC}D_2$ be optical parallelograms having the common pair of opposite corners B and C and the common centre O, then the inertia pair (O, D_1) will be said to be *congruent* to the inertia pair (O, D_2).

This will be written:

$$(O, D_1) \,(\equiv)\, (O, D_2).$$

Similarly the inertia pair (O, A_1) will be said to be *congruent* to the inertia pair (O, A_2).

If (O, D_1) be any inertia pair and a be any inertia line intersecting OD_1 in O, then the above definition enables us to show that there is one and only one element, say X, in a which is distinct from O and such that:

$$(O, D_1) \,(\equiv)\, (O, X).$$

For, by Theorem 106, there is at least one separation line, say c, which passes through O and is conjugate to both OD_1 and a.

Thus OD_1 and c determine an inertia plane, say P_1, while a and c determine an inertia plane, say P_2.

Now if D_1 be *after* O there is one single optical parallelogram in P having O as centre and D_1 as one of its corners.

If A_1 be the corner opposite D_1 and if B and C be the remaining corners, this optical parallelogram will be $A_1\overline{BC}D_1$, where B and C will lie in c.

Again in the inertia plane P_2 there will be one single optical parallelogram having B and C as a pair of opposite corners and O as centre.

If A_2 and D_2 be the remaining corners they will lie in a, and if D_2 be *after* A_2, this optical parallelogram will be $A_2\overline{BC}D_2$.

Thus we may identify D_2 with X and can say that there is *at least one* element X lying in a and distinct from O and such that:

$$(O, D_1) \,(\equiv)\, (O, X).$$

We have now to show that the element X is unique in this respect in the inertia line a.

Let c' be any other separation line distinct from c which passes through O and is conjugate to both OD_1 and a.

Then OD_1 and c' determine an inertia plane, say P_1', while a and c' determine an inertia plane, say P_2'.

There is one single optical parallelogram in P_1', having A_1 and D_1 as a pair of opposite corners, and this optical parallelogram has also O as its centre.

If B' and C' be the remaining corners this optical parallelogram will be $A_1\overline{B'C'}D_1$.

But now we have the optical parallelograms $A_1\overline{B'C'}D_1$, $A_1\overline{BC}D_1$, $A_2\overline{BC}D_2$ and the diagonal line A_2D_2 of the last of these is conjugate to $B'C'$ and so it follows, by Theorem 170, that the elements A_2, B', D_2, C' form the corners of a fourth optical parallelogram $A_2\overline{B'C'}D_2$.

Now $A_2\overline{B'C'}D_2$ will lie in the inertia plane P_2' and will have O as its centre, and further $A_2\overline{B'C'}D_2$ is the only optical parallelogram which lies in P_2' and has B' and C' as a pair of opposite corners.

Thus the element D_2 or X is independent of the particular separation line passing through O and conjugate to both OD_1 and a, which we may select as the separation diagonal line of our optical parallelograms.

It follows that there is one and only one element X in a which is such that:

$$(O, D_1)\,(\equiv)\,(O, X).$$

The same result follows if D_1 be *before* O instead of *after* it.

Again, if (O, D_1), (O, D_2) and (O, D_3) be inertia pairs such that:

$$(O, D_1)\,(\equiv)\,(O, D_2)$$

and

$$(O, D_2)\,(\equiv)\,(O, D_3),$$

we may easily show that:

$$(O, D_1)\,(\equiv)\,(O, D_3).$$

In order to see this we have only to remember that whether the inertia lines OD_1, OD_2, OD_3 all lie in one inertia plane or in one inertia threefold, there must be at least one general line passing through O and normal to all three.

Since only a separation line can be normal to an inertia line, this separation line will be conjugate to OD_1, OD_2 and OD_3, and if we call it c, then OD_1 and c will determine an inertia plane, say P_1, OD_2 and c will determine an inertia plane, say P_2, and OD_3 and c will determine an inertia plane, say P_3.

Now in P_1 there will be one single optical parallelogram having O as centre and D_1 as one of its corners, while in P_2 there will be one single optical parallelogram having O as centre and D_2 as one of its corners, and finally in P_3 there will be one single optical parallelogram having O as centre and D_3 as one of its corners.

Since $(O, D_1)(\equiv)(O, D_2)$ and $(O, D_2)(\equiv)(O, D_3)$ these three optical parallelograms will have a common pair of opposite corners, and so it follows from the definition that:

$$(O, D_1)(\equiv)(O, D_3).$$

Thus *for inertia pairs having a self-corresponding element, the relation of congruence is a transitive relation.*

It is to be observed that if (O, D_1) be an inertia pair we may write:

$$(O, D_1)(\equiv)(O, D_1),$$

or an inertia pair is to be regarded as congruent to itself.

We shall next consider the congruence of separation pairs having a self-corresponding element.

This case differs somewhat from the one we have considered.

While two intersecting inertia lines always lie in an inertia plane, two intersecting separation lines may lie either in a separation plane, an optical plane, or an inertia plane.

An inertia line can only be conjugate to two intersecting separation lines if these lie in a separation plane, as follows from Theorem 99.

Thus if we were to give a definition of the congruence of separation pairs having a self-corresponding element which was strictly analogous to that given for inertia pairs, such a definition would be incomplete.

It is however possible, by a slight modification, to give a definition which will hold for all cases.

In order to avoid complication we shall first explain what we mean by an inertia pair being "conjugate" to a separation pair or a separation pair being "conjugate" to an inertia pair.

Definition. If $A\overline{BC}D$ be an optical parallelogram and O be its centre, then the inertia pairs (O, D) and (O, A) will be spoken of as *conjugates* to the separation pairs (O, B) and (O, C) and also conversely.

The pair (O, D) will be called an *after-conjugate* to the pairs (O, B), (O, C), while (O, A) will be called a *before-conjugate* to the pairs (O, B), (O, C).

Further, either of the separation pairs (O, B), (O, C) will be called an *after-conjugate* to (O, A) and a *before-conjugate* to (O, D).

Now we know that there are an infinite number of inertia planes which contain any given separation line, and so there are always inertia pairs which are conjugate to any given separation pair.

Knowing this we can give the following definition of the "congruence" of separation pairs having a self-corresponding element.

Definition. If (O, B_1) and (O, B_2) be separation pairs and if (O, D_1) and (O, D_2) be inertia pairs which are after-conjugates to (O, B_1) and (O, B_2) respectively, then if $(O, D_1)\,(\equiv)\,(O, D_2)$ we shall say that (O, B_1) is *congruent* to (O, B_2) and shall write this:

$$(O, B_1)\{\equiv\}(O, B_2).$$

If (O, D_1') be any inertia pair which is an after-conjugate to (O, B_1), but is distinct from (O, D_1), then it is obvious by definition that:

$$(O, D_1)\,(\equiv)\,(O, D_1').$$

But since $$(O, D_1)\,(\equiv)\,(O, D_2),$$

and, since these are inertia pairs, it follows that:

$$(O, D_1')\,(\equiv)\,(O, D_2).$$

Thus the congruence of (O, B_1) to (O, B_2) is independent of the particular after-conjugate to (O, B_1) which we may select, and similarly, it is independent of the particular after-conjugate to (O, B_2) which we may select.

Again, if (O, B_1), (O, B_2) and (O, B_3) be separation pairs such that:

$$(O, B_1)\{\equiv\}(O, B_2),$$

and $$(O, B_2)\{\equiv\}(O, B_3),$$

we may easily show that:

$$(O, B_1)\{\equiv\}(O, B_3).$$

In order to prove this, let (O, D_1), (O, D_2) and (O, D_3) be inertia pairs which are after-conjugates to (O, B_1), (O, B_2) and (O, B_3) respectively.

Then we must have:

$$(O, D_1)\,(\equiv)\,(O, D_2),$$

and $$(O, D_2)\,(\equiv)\,(O, D_3),$$

and, since these are inertia pairs, it follows, as previously shown, that:

$$(O, D_1)\,(\equiv)\,(O, D_3).$$

Thus, by the definition:

$$(O, B_1)\{\equiv\}(O, B_3),$$

and so, *for separation pairs having a self-corresponding element, the relation of congruence is a transitive relation.*

Again, if (O, B) be any separation pair and a be any separation line passing through O, there are two and only two elements, say X_1 and Y_1, in a which are distinct from O and such that:

$$(O, B)\{\equiv\}(O, X_1),$$

and $$(O, B)\{\equiv\}(O, Y_1).$$

This may be easily shown as follows.

Let (O, D) be any inertia pair which is an after-conjugate to (O, B) and let b be any inertia line which passes through O and is conjugate to a.

Then, as we have already seen, there is one and only one element, say D_1, lying in b and distinct from O and such that:

$$(O, D)(\equiv)(O, D_1).$$

But now a and b determine an inertia plane and in this inertia plane there is one and only one optical parallelogram having O as centre and D_1 as one of its corners.

If this optical parallelogram be $A_1B_1C_1D_1$, then the elements B_1 and C_1 will lie in a and the inertia pair (O, D_1) will be an after-conjugate to each of the separation pairs (O, B_1) and (O, C_1).

Thus since $(O, D)(\equiv)(O, D_1)$ it follows that:

$$(O, B)\{\equiv\}(O, B_1)$$

and $$(O, B)\{\equiv\}(O, C_1).$$

Again, if there were any other element, say B_2, lying in a and distinct from both B_1 and C_1 and such that we had

$$(O, B)\{\equiv\}(O, B_2),$$

then there would be an element, say D_2, lying in b and such that (O, D_2) was an after-conjugate to (O, B_2).

Since B_2 is supposed distinct from both B_1 and C_1, therefore D_2 would require to be distinct from D_1.

But since we have supposed $(O, B)\{\equiv\}(O, B_2)$, therefore we should have $(O, D)(\equiv)(O, D_2)$ and so we should have the two distinct elements D_1 and D_2 lying in the inertia line b and such that:

$$(O, D)(\equiv)(O, D_1) \quad \text{and} \quad (O, D)(\equiv)(O, D_2),$$

which we have already shown to be impossible.

Thus we may identify B_1 with X_1 and C_1 with Y_1 and say that there are two and only two elements X_1 and Y_1 lying in a and distinct from O and such that:

$$(O, B)\{\equiv\}(O, X_1) \quad \text{and} \quad (O, B)\{\equiv\}(O, Y_1).$$

If $A\overline{BCD}$ be an optical parallelogram and O be its centre, we observe that according to our definitions we have

$$(O, B)\{\equiv\}(O, C),$$

but not

$$(O, A)(\equiv)(O, D).$$

The reason why we make this distinction is that in the separation pairs we have O neither *before* nor *after* B and also O neither *before* nor *after* C, while in the inertia pairs we have O *after* A and O *before* D.

Thus in the first case the relations are alike in respect of *before* and *after*, while in the second case the relations are different.

The question now arises as to the "congruence" of optical pairs.

In this case constructions such as those by which we defined the congruence of inertia and separation pairs having a self-corresponding element, entirely fail and there is nothing at all analogous to them.

We are thus led to regard optical pairs as not determinately comparable with one another in respect of congruence, except when they lie in the same, or in parallel, optical lines.

As regards the "congruence" of pairs lying in the same general line, we have as yet given no definition, except for the very special case of inertia or separation pairs having a self-corresponding element; while no definition whatever has been given of the "congruence" of pairs lying in parallel general lines.

A definition covering all these omitted cases can be given, which applies to all three types of pair.

We must first however define what we mean when we say that one pair is opposite to another.

Definition. A pair (A, B) will be said to be *opposite* to a pair (C, D) if and only if the elements A, B, C, D form the corners of a general parallelogram in such a way that AB and CD are one pair of opposite sides, while AC and BD are the other pair of opposite sides.

This will be denoted by the symbols

$$(A, B)\,\square\,(C, D).$$

It will be observed that the use of the symbol $\square$ implies that the pairs (A, B) and (C, D) lie in distinct general lines which are parallel to one another.

If however we have

$$(A, B)\,\square\,(C, D),$$

and

$$(E, F)\,\square\,(C, D),$$

then the pairs (A, B) and (E, F) may lie either in the same or in parallel general lines.

If (A, B) and (E, F) do not lie in the same general line, it follows from Theorem 126 that we may write

$$(A, B) \square (E, F).$$

We have now to prove the following theorem:

THEOREM 171

If (A, B), (A', B') *and* (C, D) *be pairs such that:*

$$(A, B) \square (C, D),$$

and
$$(A', B') \square (C, D),$$

and if (C', D') *be any other pair such that:*

$$(A, B) \square (C', D'),$$

and which does not lie in the general line $A'B'$, *then we shall also have*

$$(A', B') \square (C', D').$$

We shall first consider the case where (A, B) and (A', B') do not lie in one general line.

In this case since

$$(A, B) \square (C, D),$$

and
$$(A', B') \square (C, D),$$

it follows, by Theorem 126, that:

$$(A', B') \square (A, B).$$

But
$$(C', D') \square (A, B)$$

by hypothesis, and so, since (C', D') and (A', B') do not lie in one general line, it follows that:

$$(A', B') \square (C', D').$$

Next consider the case where (A, B) and (A', B') lie in one general line.

There are two sub-cases of this:

(1) (C, D) and (C', D') do not lie in one general line.

(2) (C, D) and (C', D') do lie in one general line.

Consider first sub-case (1).

Here since
$$(C, D) \square (A, B),$$

and
$$(C', D') \square (A, B),$$

and since (C, D) and (C', D') do not lie in one general line, it follows that:

$$(C', D') \square (C, D).$$

But
$$(A', B') \square (C, D),$$

and so, since (C', D') and (A', B') do not lie in one general line, it follows that:

$$(A', B') \square (C', D').$$

Next consider sub-case (2).

Let E be any element in the general line AC' distinct from both A and C' and let a general line through E parallel to AB intersect $D'B$ in the element F.

Then we shall have

$$(E, F) \square (A, B),$$

and also

$$(E, F) \square (C', D').$$

But now, since E is distinct from A and also from C', it follows that the general line EF must be distinct from the general line containing (A, B) and (A', B') and must also be distinct from the general line containing (C', D') and (C, D).

Thus since

$$(E, F) \square (A, B),$$

and

$$(C, D) \square (A, B),$$

and, since (E, F) and (C, D) do not lie in one general line, it follows that:

$$(E, F) \square (C, D).$$

Also since

$$(A', B') \square (C, D),$$

and, since (E, F) and (A', B') do not lie in one general line, it follows that:

$$(A', B') \square (E, F).$$

But

$$(C', D') \square (E, F),$$

and since (A', B') and (C', D') lie respectively in the distinct general lines AB and CD, it follows that:

$$(A', B') \square (C', D').$$

Thus the theorem holds in all cases.

We are now in a position to introduce the following definition:

Definition. A pair (A, B) will be said to be *co-directionally congruent* to a pair (A', B') provided a pair (C, D) exists such that:

$$(A, B) \square (C, D),$$

and

$$(A', B') \square (C, D).$$

The theorem just proved shows that we are at liberty to replace the pair (C, D) by any other pair (C', D') such that:

$$(A, B) \square (C', D'),$$

provided (C', D') does not lie in the general line $A'B'$.

It is evident that $(A, B) \square (C, D)$ implies that (A, B) is co-directionally congruent to (C, D), but (A, B) being co-directionally congruent to (C, D) does not imply that $(A, B) \square (C, D)$, since (A, B) and (C, D) might lie in the same general line.

It is also obvious that (A, B) is co-directionally congruent to (A, B).

We shall ultimately represent co-directional congruence by the same symbol $\equiv$ as we shall use for the other cases of congruence, but when we wish to make it clear that the congruence is co-directional we shall use the symbol $|\equiv|$.

Thus we see that: $(A, B) \square (C, D)$ implies $(A, B) |\equiv| (C, D)$, but $(A, B) |\equiv| (C, D)$ does not imply $(A, B) \square (C, D)$, except when AB and CD are distinct general lines.

We have next to show that if

$$(A, B) |\equiv| (C, D),$$

and

$$(C, D) |\equiv| (E, F),$$

then must

$$(A, B) |\equiv| (E, F).$$

This is easily proved; for if a be any general line parallel to AB but distinct from CD and EF and therefore also parallel to them, we may select any pair (G, H) in a, such that:

$$(A, B) \square (G, H) \quad \text{.........................(1).}$$

Then, since

$$(A, B) |\equiv| (C, D),$$

it follows, by Theorem 171, that:

$$(C, D) \square (G, H).$$

Similarly since

$$(C, D) |\equiv| (E, F),$$

it follows that:

$$(E, F) \square (G, H) \quad \text{.........................(2).}$$

Thus from (1) and (2) it follows that:

$$(A, B) |\equiv| (E, F),$$

and so we see that: *the relation of co-directional congruence of pairs is a transitive relation.*

If $(A, B) \square (C, D)$ and if B be *after* A then it is easy to see that D must be *after* C.

In the first place AB must be either an optical or inertia line and, since CD is parallel to AB, it follows that CD must be the same type of general line as AB.

Suppose first that AB is an optical line.

Then C could not be *after* D, for then, by Theorem 58 or Theorem 92, AC and BD would intersect, contrary to the hypothesis that they are parallel.

Thus, since C and D are distinct, and since CD is an optical line, it follows that D must be *after* C.

Next suppose that AB is an inertia line.

Then AB and CD must lie in an inertia plane, say P.

If AC and BD should happen to be optical lines then, since B is *after* A it follows that BD would be an after-parallel of AC and so, since CD is an inertia line, it would follow that D must be *after* C.

Next suppose that AC and BD are not optical lines.

Let AE and BE be generators of P of opposite sets passing through A and B respectively and intersecting in E.

Let CF be an optical line through C parallel to AE and let it intersect the general line through E parallel to AC in F.

Then EF must be parallel to BD and so, by Theorem 126, DF must be parallel to BE and therefore must be an optical line.

But now, since B is *after* A, we must have E *after* A and B *after* E.

Thus, by the first case, we must have F *after* C and D *after* F and therefore D *after* C as was to be proved.

Thus in all cases if B be *after* A we must have D *after* C and similarly if B be *before* A we must have D *before* C.

It follows directly from this that if

$$(A, B) \square (C, D),$$

and

$$(A', B') \square (C, D),$$

then if B be *after* A we must have D *after* C and therefore B' *after* A'.

Thus *if* $(A, B) \mid\equiv\mid (A', B')$ *and if* B *be* after A *we must have* B' after A', *while if* B *be* before A *we must have* B' before A'.

Again if three corners of a general parallelogram A', C and D be given and if we know that two of the side lines are $A'C$ and CD, then the general parallelogram is uniquely determined.

If then any pair (A', X) be co-directionally congruent to a pair (A, B), where A, B and A' are given, it is easy to see that X is uniquely determinate, provided we know that A' corresponds to A.

For let a be a general line parallel to AB, but which does not pass through A', and let (C, D) be any pair in a such that:

$$(A, B) \square (C, D).$$

Then there is one single general parallelogram having A', C and D as three of its corners and $A'C$ and CD as two of its side lines.

If B' be the remaining corner we shall have

$$(A', B') \square (C, D),$$

and so

$$(A', B') \mid\equiv\mid (A, B).$$

Thus X must be identified with B', which is a definite element.

THEOREM 172

If (O_1, A_1) and (O_2, A_2) be inertia pairs such that:

$$(O_1, A_1) \mid \equiv \mid (O_2, A_2),$$

and if (O_1, B_1) be any separation pair which is conjugate to (O_1, A_1), then there is a separation pair, say (O_2, B_2), which is conjugate to (O_2, A_2), and such that:

$$(O_1, B_1) \mid \equiv \mid (O_2, B_2).$$

It is evident that (O_1, A_1) and (O_1, B_1) must lie in an inertia plane, say P_1.

Since the inertia line O_2A_2 must be either parallel to the inertia line O_1A_1, or else identical with it, it follows that O_2A_2 must either lie in P_1 or in an inertia plane parallel to P_1.

We shall first consider the case where O_2A_2 lies in an inertia plane P_2 parallel to P_1.

If now we take the one single optical parallelogram in P_1 having O_1 as centre and A_1 as one of its corners, then B_1 will be another corner.

If (O_1, B_1) be an after-conjugate to (O_1, A_1) we may take this optical parallelogram to be $A_1\overline{B_1C_1}D_1$, while if (O_1, B_1) be a before-conjugate to (O_1, A_1) we may take the optical parallelogram to be $D_1\overline{B_1C_1}A_1$.

Now the inertia plane P_1 and the general line O_1O_2 determine a general threefold containing P_2 and so, as we have already seen, if through any element of P_1 distinct from O_1 a general line be taken parallel to O_1O_2, then this general line will intersect P_2.

Now through the elements A_1, B_1, C_1 and D_1 let general lines be taken parallel to O_1O_2 and let these intersect P_2 in the elements A_2, B_2, C_2 and D_2 respectively.

Then any two of the general lines O_1O_2, A_1A_2, B_1B_2, C_1C_2, D_1D_2 are parallel to one another and therefore any two of them lie in a general plane.

Since however the elements A_1, O_1 and D_1 lie in one general line, the three general lines A_1A_2, O_1O_2 and D_1D_2 lie in one general plane, and since the elements B_1, O_1 and C_1 lie in one general line, the three general lines B_1B_2, O_1O_2 and C_1C_2 lie in one general plane.

Thus the elements A_2, O_2 and D_2 lie in one general line parallel to the general line containing A_1, O_1 and D_1, while the elements B_2, O_2 and C_2 lie in another general line parallel to that containing B_1, O_1 and C_1.

Further the general lines A_2B_2, A_2C_2, B_2D_2, C_2D_2 must be respectively parallel to A_1B_1, A_1C_1, B_1D_1, C_1D_1 and, since these latter

are all optical lines, it follows that A_2B_2, A_2C_2, B_2D_2, C_2D_2 are all optical lines.

Thus A_2, B_2, C_2, D_2 form the corners of an optical parallelogram having O_2 as centre.

Further, the diagonal line A_2D_2 is an inertia line, while the diagonal line B_2C_2 must be a separation line.

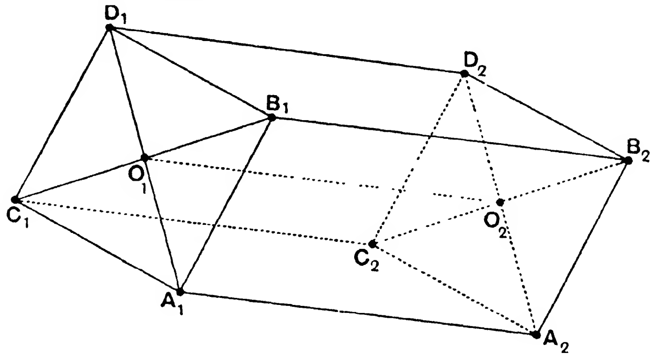


Fig. 43.

Thus the separation pair (O_2, B_2) is conjugate to the inertia pair (O_2, A_2), and, since O_2 is *after* or *before* A_2 according as O_1 is *after* or *before* A_1, it follows that (O_2, B_2) is an after- or before-conjugate to (O_2, A_2) according as (O_1, B_1) is an after- or before-conjugate to (O_1, A_1).

Also we have $\qquad (O_1, B_1) \square (O_2, B_2)$

and so $\qquad (O_1, B_1) \mid \equiv \mid (O_2, B_2)$.

This proves the theorem provided O_2A_2 does not lie in P_1.

Consider next the case where O_2A_2 does lie in P_1.

Let P' be any inertia plane parallel to P_1 and let (O', A') be any inertia pair in P' such that:

$$(O_1, A_1) \square (O', A').$$

Then, by Theorem 171, since $(O_1, A_1) \mid \equiv \mid (O_2, A_2)$, we must have

$$(O_2, A_2) \square (O', A').$$

Thus, by the case already proved, it follows that there is an optical parallelogram lying in P' which has O' as centre and A' as one of its corners and such that, if we denote it by $A'\overline{B'C'}D'$ or $D'\overline{B'C'}A'$ (according as the optical parallelogram in P_1 is $A_1\overline{B_1C_1}D_1$ or $D_1\overline{B_1C_1}A_1$), then:

$$(O_1, B_1) \square (O', B').$$

But in a similar manner we can show that there is an optical parallelogram lying in P_1 which has O_2 as centre and A_2 as one of its corners and

such that if we denote it by $A_2\overline{B_2C_2}D_2$ or $D_2\overline{B_2C_2}A_2$ (according as the optical parallelogram in P' is $A'\overline{B'C'}D'$ or $D'\overline{B'C'}A'$), then:

$$(O_2, B_2) \square (O', B').$$

Thus it follows from definition that:

$$(O_1, B_1) \mid \equiv \mid (O_2, B_2).$$

Further, (O_2, B_2) is conjugate to (O_2, A_2) and will be an after- or before-conjugate to (O_2, A_2) according as (O_1, B_1) is an after- or before-conjugate to (O_1, A_1).

Thus the theorem holds in all cases.

Theorem 173

If (O_1, A_1) and (O_2, A_2) be separation pairs such that:

$$(O_1, A_1) \mid \equiv \mid (O_2, A_2),$$

and if (O_1, B_1) be any inertia pair which is conjugate to (O_1, A_1), then there is an inertia pair, say (O_2, B_2), which is conjugate to (O_2, A_2) and such that:

$$(O_1, B_1) \mid \equiv \mid (O_2, B_2).$$

The proof of this theorem is quite analogous to that of Theorem 172.

Also it will be seen that (O_2, B_2) will be a before- or after-conjugate to (O_2, A_2) according as (O_1, B_1) is a before- or after-conjugate to (O_1, A_1).

We have now to prove certain theorems involving both the co-directional congruence of pairs and the congruence of pairs having a self-corresponding element.

We shall make use of the symbols $(\equiv)$, $\{\equiv\}$ and $\mid \equiv \mid$ in the manner already explained in order to show clearly the types of congruence to which we refer.

Theorem 174

If (O_1, A_1), (O_1, B_1) and (O_2, A_2) be inertia pairs such that:

$$(O_1, A_1) (\equiv) (O_1, B_1)$$

and

$$(O_1, A_1) \mid \equiv \mid (O_2, A_2),$$

then there is an inertia pair (O_2, B_2) such that:

$$(O_2, A_2) (\equiv) (O_2, B_2)$$

and

$$(O_1, B_1) \mid \equiv \mid (O_2, B_2).$$

Let c be any separation line passing through O_1 and normal to both the inertia lines O_1A_1 and O_1B_1, and let C_1 be an element in c such that the separation pair (O_1, C_1) is conjugate to (O_1, A_1).

Then, since $(O_1, A_1)\,(\equiv)\,(O_1, B_1)$, it follows that (O_1, C_1) must also be conjugate to (O_1, B_1).

But, since $(O_1, A_1) \mid\equiv\mid (O_2, A_2)$, it follows, by Theorem 172, that there is a separation pair, say (O_2, C_2), which is conjugate to (O_2, A_2) and such that:

$$(O_1, C_1) \mid\equiv\mid (O_2, C_2).$$

But now, by Theorem 173, since (O_1, B_1) is conjugate to (O_1, C_1), it follows that there is an inertia pair, say (O_2, B_2), which is conjugate to (O_2, C_2) and such that:

$$(O_1, B_1) \mid\equiv\mid (O_2, B_2).$$

But now, since $(O_1, A_1)\,(\equiv)\,(O_1, B_1)$ and these are inertia pairs, we must have A_1 and B_1 either both *after* O_1 or both *before* O_1.

Further, A_2 must be *after* or *before* O_2 according as A_1 is *after* or *before* O_1, while B_2 must be *after* or *before* O_2 according as B_1 is *after* or *before* O_1.

Thus A_2 and B_2 are either both *after* O_2 or both *before* O_2.

Since therefore (O_2, C_2) is conjugate to both (O_2, A_2) and (O_2, B_2), it follows that:

$$(O_2, A_2)\,(\equiv)\,(O_2, B_2).$$

Thus the theorem is proved.

THEOREM 175

If (O_1, A_1), (O_1, B_1) and (O_2, A_2) be separation pairs such that:

$$(O_1, A_1)\,\{\equiv\}\,(O_1, B_1)$$

and

$$(O_1, A_1) \mid\equiv\mid (O_2, A_2),$$

then there is a separation pair (O_2, B_2) such that:

$$(O_2, A_2)\,\{\equiv\}\,(O_2, B_2)$$

and

$$(O_1, B_1) \mid\equiv\mid (O_2, B_2).$$

Let (O_1, D_1) and (O_1, E_1) be inertia pairs which are after-conjugates to (O_1, A_1) and (O_1, B_1) respectively.

Then since $\quad (O_1, A_1)\,\{\equiv\}\,(O_1, B_1),$

we must have $\quad (O_1, D_1)\,(\equiv)\,(O_1, E_1).$

But now since $\quad (O_1, A_1) \mid\equiv\mid (O_2, A_2),$

and since (O_1, D_1) is an inertia pair which is an after-conjugate to (O_1, A_1), it follows, by Theorem 173, that there is an inertia pair, say (O_2, D_2), which is an after-conjugate to (O_2, A_2) and such that:

$$(O_1, D_1) \mid\equiv\mid (O_2, D_2).$$

But now (O_1, D_1), (O_1, E_1) and (O_2, D_2) are inertia pairs such that:

$$(O_1, D_1)\,(\equiv)\,(O_1, E_1)$$

and
$$(O_1, D_1)\,|\equiv|\,(O_2, D_2),$$

and so, by Theorem 174, there is an inertia pair (O_2, E_2) such that:

$$(O_2, D_2)\,(\equiv)\,(O_2, E_2)$$

and
$$(O_1, E_1)\,|\equiv|\,(O_2, E_2).$$

Since however (O_1, B_1) is a separation pair which is a before-conjugate to the inertia pair (O_1, E_1), it follows, by Theorem 172, that there is a separation pair, say (O_2, B_2), which is a before-conjugate to (O_2, E_2) and such that:

$$(O_1, B_1)\,|\equiv|\,(O_2, B_2).$$

But since (O_2, D_2) and (O_2, E_2) are after-conjugates to (O_2, A_2) and (O_2, B_2) respectively, and since

$$(O_2, D_2)\,(\equiv)\,(O_2, E_2),$$

it follows by definition that:

$$(O_2, A_2)\,\{\equiv\}\,(O_2, B_2).$$

Thus the theorem is proved.

THEOREM 176

If (A, B) and (A, C) be inertia pairs such that:

$$(A, B)\,(\equiv)\,(A, C),$$

then there is an inertia pair (C, D) such that:

$$(C, A)\,(\equiv)\,(C, D)$$

and
$$(B, A)\,|\equiv|\,(C, D).$$

Let a be any separation line which passes through A and is normal to both AB and AC.

Let A_1 be an element in a such that the separation pair (A, A_1) is conjugate to the inertia pair (A, C).

Then since
$$(A, B)\,(\equiv)\,(A, C),$$

it follows that (A, A_1) is also conjugate to (A, B).

Let the general line through C parallel to AA_1 intersect the general line through A_1 parallel to AC in the element C_1, and let the general line through B parallel to AA_1 intersect the general line through A_1 parallel to AB in the element B_1.

Thus $(C, C_1) \square (A, A_1)$
and $(B, B_1) \square (A, A_1)$,
and therefore $(B, B_1) \mid\equiv\mid (C, C_1)$.

Now, since (A, A_1) is conjugate to (A, C), it follows that A_1C is an optical line and it is easy to show that AC_1 is also an optical line as follows:

Since A_1C and AC_1 are diagonal lines of the general parallelogram whose corners are A, A_1, C_1, C, it follows that they intersect in an element, say E, which is the mean of A_1 and C.

If F be the mean of A_1 and A, then EF is parallel to CA and therefore EF is conjugate to the separation line AA_1.

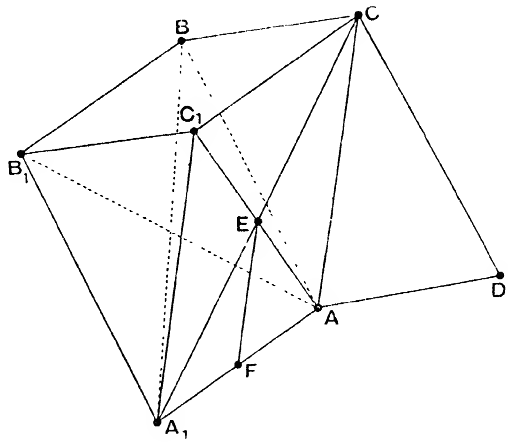


Fig. 44.

Thus A, A_1 and E are three corners of an optical parallelogram whose centre F lies in AA_1 and therefore AE (that is AC_1) is an optical line.

Similarly, since (A, A_1) is conjugate to (A, B), it follows that AB_1 is an optical line.

But since CC_1 and BB_1 are parallel to AA_1 we have CC_1 conjugate to CA, and BB_1 conjugate to BA.

Thus the inertia pairs (C, A) and (B, A) are conjugate to the separation pairs (C, C_1) and (B, B_1) respectively.

But now, since $(B, B_1) \mid\equiv\mid (C, C_1)$,
and since (B, A) is an inertia pair which is conjugate to (B, B_1), it follows, by Theorem 173, that there is an inertia pair, say (C, D), which is conjugate to (C, C_1) and such that:

$$(B, A) \mid\equiv\mid (C, D).$$

But from this it follows that D must be *before* or *after* C according as A is *before* or *after* B.

Since however $$(A, B)(\equiv)(A, C),$$
we have A *before* or *after* B according as A is *before* or *after* C.

Thus we must have D *before* or *after* C according as A is *before* or *after* C.

Since therefore the inertia pairs (C, A) and (C, D) are both conjugate to the separation pair (C, C_1), it follows that:
$$(C, A)(\equiv)(C, D).$$

Thus the theorem is proved.

THEOREM 177

If (A, B) and (A, C) be separation pairs such that:
$$(A, B)\{\equiv\}(A, C),$$
then there is a separation pair (C, D) such that:
$$(C, A)\{\equiv\}(C, D)$$
and $$(B, A)|\equiv|(C, D).$$

Let a be any inertia line which passes through A and is normal to AB, and let a' be any inertia line which passes through A and is normal to AC.

Let A_1 be an element in a such that the inertia pair (A, A_1) is an after-conjugate to the separation pair (A, B) and let A' be an element in a' such that the inertia pair (A, A') is an after-conjugate to the separation pair (A, C).

Then since $$(A, B)\{\equiv\}(A, C)$$
it follows that: $$(A, A_1)(\equiv)(A, A').$$

Let the general line through C parallel to AA' intersect the general line through A' parallel to AC in the element C' and let the general line through B parallel to AA_1 intersect the general line through A_1 parallel to AB in the element B_1.

Then $$(C, C')\square(A, A')$$
and $$(B, B_1)\square(A, A_1),$$
and so we may write: $$(C, C')|\equiv|(A, A'),$$
and $$(B, B_1)|\equiv|(A, A_1).$$

But, since (A, A'), (A, A_1) and (C, C') are inertia pairs such that
$$(A, A')(\equiv)(A, A_1)$$
and $$(A, A')|\equiv|(C, C'),$$

therefore, by Theorem 174, there is an inertia pair (C, C_1) such that:

$$(C, C')\,(\equiv)\,(C, C_1)$$

and $$(A, A_1) \mid \equiv \mid (C, C_1).$$

Thus since $$(B, B_1) \mid \equiv \mid (A, A_1),$$

it follows that: $$(B, B_1) \mid \equiv \mid (C, C_1).$$

But now we may show in the manner employed in the last theorem that, since (A, A_1) is an after-conjugate to (A, B), therefore (B, B_1) is an after-conjugate to (B, A), and, since (A, A') is an after-conjugate to (A, C), therefore (C, C') is an after-conjugate to (C, A).

Since, however, we have the inertia pairs (B, B_1) and (C, C_1) such that:

$$(B, B_1) \mid \equiv \mid (C, C_1),$$

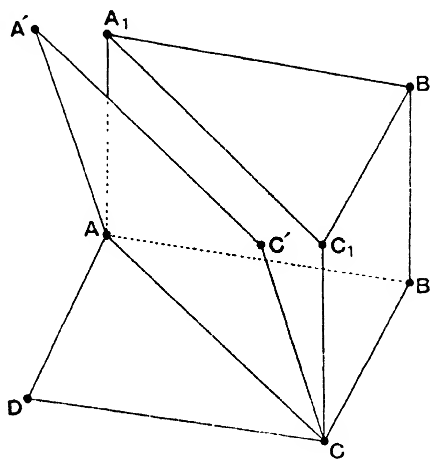


Fig. 45.

and, since (B, A) is a separation pair which is conjugate to (B, B_1), it follows, by Theorem 172, that there is a separation pair, say (C, D), which is conjugate to (C, C_1) and such that:

$$(B, A) \mid \equiv \mid (C, D).$$

But now (A, A_1) is an after-conjugate to (A, B) and so A_1 is *after* A.

Thus, since $$(A, A_1) \mid \equiv \mid (C, C_1),$$

it follows that C_1 is *after* C, and so, since (C, C_1) is conjugate to (C, D), it must be an after-conjugate.

But (C, C') is an after-conjugate to (C, A) and so since

$$(C, C')\,(\equiv)\,(C, C_1),$$

it follows from the definition that:

$$(C, A)\{\equiv\}(C, D).$$

Thus the theorem is proved.

THEOREM 178

(1) *If A, B and C be three distinct elements and the pairs (A, B) and (B, C) be such that:*

$$(A, B) \mid\equiv\mid (B, C),$$

then B is the mean of A and C.

(2) *If A, B and C be three distinct elements such that B is the mean of A and C, then the pairs (A, B) and (B, C) are such that:*

$$(A, B) \mid\equiv\mid (B, C).$$

First suppose that: $(A, B) \mid\equiv\mid (B, C)$.

Then, by the definition of co-directional congruence, there must be a pair, say (D, E), such that:

$$(A, B) \square (D, E)$$

and

$$(B, C) \square (D, E).$$

Now, since the pairs (A, B) and (B, C) have a common element B, they cannot lie in parallel general lines and so must lie in the same general line.

Then BE and CD must be the diagonal lines of the general parallelogram whose corners are B, C, D and E and so BE and CD must intersect in an element F which is the mean of D and C.

But D does not lie in the general line AC, and so, since BF is parallel to AD, it follows, by Theorem 78, 94, or 118, that B is the mean of A and C.

Next, to prove the second part of the theorem, suppose that B is the mean of A and C.

Let (D, E) be any pair such that:

$$(B, C) \square (D, E).$$

Then the diagonal lines BE and CD of the general parallelogram, whose corners are B, C, D and E, must intersect in an element F which is the mean of D and C.

But, since D does not lie in the general line AC, it follows, by corollary to Theorem 78, 94 or 118, that BF (that is BE) is parallel to AD.

Thus, since also AB is parallel to DE, it follows that:

$$(A, B) \square (D, E).$$

Thus, by the definition of co-directional congruence, we have:

$$(A, B) \mid\equiv\mid (B, C).$$

Thus both parts of the theorem are proved.

We are now in a position to introduce general definitions of the congruence of inertia and separation pairs.

This is done by combining co-directional congruence with congruence in which an element is self-corresponding, in the following manner.

Definition. An inertia pair (A_1, B_1) will be said to be *congruent* to an inertia pair (A_2, B_2) provided an inertia pair (A_2, C_2) exists such that:

$$(A_1, B_1) \mid\equiv\mid (A_2, C_2)$$

and $$(A_2, B_2) (\equiv) (A_2, C_2).$$

Definition. A separation pair (A_1, B_1) will be said to be *congruent* to a separation pair (A_2, B_2) provided a separation pair (A_2, C_2) exists such that:

$$(A_1, B_1) \mid\equiv\mid (A_2, C_2)$$

and $$(A_2, B_2) \{\equiv\} (A_2, C_2).$$

We shall denote the generalized congruence of inertia or of separation pairs by the symbol $\equiv$, thus:

$$(A_1, B_1) \equiv (A_2, B_2).$$

We shall also use the same symbol to denote the congruence of optical pairs, but, in the latter case, it is to be regarded as simply equivalent to the symbol $\mid\equiv\mid$, since the only congruence of optical pairs is taken to be co-directional.

Let us consider now two inertia pairs (A_1, B_1) and (A_2, B_2) such that:

$$(A_1, B_1) \equiv (A_2, B_2).$$

Then there exists an inertia pair (A_2, C_2) such that:

$$(A_1, B_1) \mid\equiv\mid (A_2, C_2)$$

and $$(A_2, B_2) (\equiv) (A_2, C_2).$$

But, by Theorem 174, there exists an inertia pair (A_1, C_1) such that:

$$(A_2, B_2) \mid\equiv\mid (A_1, C_1)$$

and $$(A_1, B_1) (\equiv) (A_1, C_1).$$

Thus we may write $$(A_2, B_2) \equiv (A_1, B_1).$$

Again, by Theorem 176, there is an inertia pair (B_2, D_2) such that:

$$(B_2, A_2) (\equiv) (B_2, D_2)$$

and $$(C_2, A_2) \mid\equiv\mid (B_2, D_2).$$

Since however $$(C_2, A_2) \mid\equiv\mid (B_1, A_1),$$

we have $$(B_1, A_1) \mid\equiv\mid (B_2, D_2),$$

which together with the relation

$$(B_2, A_2) (\equiv) (B_2, D_2)$$

gives us
$$(B_1, A_1) \equiv (B_2, A_2).$$

If now we take instead two separation pairs (A_1, B_1) and (A_2, B_2) such that:

$$(A_1, B_1) \equiv (A_2, B_2),$$

then by using Theorem 175 in place of Theorem 174, we may prove that:

$$(A_2, B_2) \equiv (A_1, B_1).$$

Also, by a similar method to that employed in the case of inertia pairs, but using Theorem 177 in place of Theorem 176, we may prove that:

$$(B_1, A_1) \equiv (B_2, A_2).$$

Again, if (A, B) be an inertia pair, we have

$$(A, B) \mid\equiv\mid (A, B)$$

and
$$(A, B) (\equiv) (A, B).$$

Thus we have
$$(A, B) \equiv (A, B).$$

A similar result obviously holds if (A, B) be a separation pair.

Again if (A, B) and (A, C) be inertia pairs such that:

$$(A, B) (\equiv) (A, C),$$

then since
$$(A, C) \mid\equiv\mid (A, C),$$

we may write
$$(A, B) \equiv (A, C).$$

A similar result holds if (A, B) and (A, C) be separation pairs such that:

$$(A, B) \{\equiv\} (A, C).$$

Further, it is also clear that:

$$(A_1, B_1) \mid\equiv\mid (A_2, B_2)$$

implies
$$(A_1, B_1) \equiv (A_2, B_2),$$

both when (A_1, B_1) and (A_2, B_2) are inertia pairs and when they are separation pairs.

Again if (A_1, B_1), (A_2, B_2) and (A_3, B_3) be inertia pairs such that:

$$(A_1, B_1) \equiv (A_2, B_2)$$

and
$$(A_2, B_2) \equiv (A_3, B_3),$$

then, by the definition of congruence, there is an inertia pair (A_2, C_2) such that:

$$(A_1, B_1) \mid\equiv\mid (A_2, C_2) \qquad \ldots\ldots(1)$$

and
$$(A_2, B_2) (\equiv) (A_2, C_2) \qquad \ldots\ldots(2).$$

Also there is an inertia pair (A_3, C_3) such that:

$$(A_2, B_2) \mid\equiv\mid (A_3, C_3) \qquad \ldots\ldots(3)$$

and $$(A_3, B_3) (\equiv) (A_3, C_3) \qquad \ldots\ldots(4).$$

Now from (2) and (3) it follows, by Theorem 174, that there is an inertia pair (A_3, D_3) such that:

$$(A_3, C_3) (\equiv) (A_3, D_3) \qquad \ldots\ldots(5)$$

and $$(A_2, C_2) \mid\equiv\mid (A_3, D_3) \qquad \ldots\ldots(6).$$

But from (1) and (6) it follows that:

$$(A_1, B_1) \mid\equiv\mid (A_3, D_3) \qquad \ldots\ldots(7),$$

while from (4) and (5) it follows that:

$$(A_3, B_3) (\equiv) (A_3, D_3) \qquad \ldots\ldots(8).$$

Thus from (7) and (8) it follows that:

$$(A_1, B_1) \equiv (A_3, B_3).$$

A similar result may be proved for the case of separation pairs; using Theorem 175 in place of Theorem 174.

Thus for inertia or separation pairs the general relation of congruence is a transitive relation.

Again, if (A, B) be any separation pair and P be any inertia plane containing the separation line AB, there is one single optical parallelogram in P having B as centre and A as one of its corners.

If C be the corner opposite to A then, by definition, B is the mean of A and C.

Thus, by Theorem 178, we have:

$$(A, B) \mid\equiv\mid (B, C).$$

But also by definition we have:

$$(B, A) \{\equiv\} (B, C).$$

And so: $$(A, B) \equiv (B, A).$$

We have not however a corresponding result in the case either of inertia or optical pairs since the elements in such pairs are asymmetrically related.

Theorem 179

If (O_1, D_1) and (O_2, D_2) be inertia pairs while (O_1, B_1) and (O_2, B_2) are separation pairs which are before-conjugates to (O_1, D_1) and (O_2, D_2) respectively or else after-conjugates to (O_1, D_1) and (O_2, D_2) respectively; then:

(1) *If* $$(O_1, D_1) \equiv (O_2, D_2)$$

we shall also have $$(O_1, B_1) \equiv (O_2, B_2).$$

(2) *If* $(O_1, B_1) \equiv (O_2, B_2)$

we shall also have $(O_1, D_1) \equiv (O_2, D_2)$.

Let us consider the first part of the theorem.

Since $(O_1, D_1) \equiv (O_2, D_2)$,

it follows by definition that there is a pair (O_2, D') such that:

$$(O_1, D_1) \mid\equiv\mid (O_2, D')$$

and $$(O_2, D_2) (\equiv) (O_2, D').$$

Then (O_2, D') is an inertia pair and so, since (O_1, B_1) is a separation pair which is conjugate to (O_1, D_1), it follows, by Theorem 172, that there is a separation pair, say (O_2, B'), which is conjugate to (O_2, D') and such that:

$$(O_1, B_1) \mid\equiv\mid (O_2, B').$$

Now if D_2 be *after* O_2 we shall also have D' *after* O_2 and so (O_2, D_2) and (O_2, D') will be after-conjugates to (O_2, B_2) and (O_2, B') respectively.

Thus we shall have

$$(O_2, B_2) \{\equiv\} (O_2, B').$$

If, on the other hand, D_2 be *before* O_2 we shall also have D' *before* O_2, and so (O_2, D_2) and (O_2, D') will be before-conjugates to (O_2, B_2) and (O_2, B') respectively.

Now by completing the optical parallelograms implied in the relation:

$$(O_2, D_2) (\equiv) (O_2, D'),$$

we see that in this case there will be inertia pairs, say (O_2, A_2) and (O_2, A') which will be after-conjugates to (O_2, B_2) and (O_2, B') respectively, and such that:

$$(O_2, A_2) (\equiv) (O_2, A').$$

Thus we have also in this case

$$(O_2, B_2) \{\equiv\} (O_2, B').$$

Combining this with the relation:

$$(O_1, B_1) \mid\equiv\mid (O_2, B'),$$

it follows by definition that:

$$(O_1, B_1) \equiv (O_2, B_2).$$

Thus the first part of the theorem is proved.

Consider now the second part of the theorem.

Since $(O_1, B_1) \equiv (O_2, B_2)$,

it follows by definition that there is a pair (O_2, B') such that:

$$(O_1, B_1) \mid\equiv\mid (O_2, B')$$

and $$(O_2, B_2) \{\equiv\} (O_2, B').$$

Then (O_2, B') is a separation pair and so, since (O_1, D_1) is an inertia pair which is conjugate to (O_1, B_1), it follows, by Theorem 173, that there is an inertia pair, say (O_2, D'), which is conjugate to (O_2, B') and such that:

$$(O_1, D_1) \mid \equiv \mid (O_2, D').$$

Now if (O_1, B_1) and (O_2, B_2) be before-conjugates to (O_1, D_1) and (O_2, D_2) respectively we shall have D_1 *after* O_1 and therefore D' *after* O_2, and also we shall have D_2 *after* O_2.

Thus (O_2, D_2) and (O_2, D') will be after-conjugates to (O_2, B_2) and (O_2, B') respectively and so, since

$$(O_2, B_2) \{\equiv\} (O_2, B'),$$

it follows that: $$(O_2, D_2) (\equiv) (O_2, D').$$

If, on the other hand, (O_1, B_1) and (O_2, B_2) be after-conjugates to (O_1, D_1) and (O_2, D_2) respectively, we shall have D_1 *before* O_1 and therefore D' *before* O_2 and also we shall have D_2 *before* O_2.

Thus (O_2, D_2) and (O_2, D') will be before-conjugates to (O_2, B_2) and (O_2, B') respectively.

Now by completing the optical parallelograms implied in these relations we see that there are inertia pairs, say (O_2, A_2) and (O_2, A'), which are after-conjugates to (O_2, B_2) and (O_2, B') respectively and such that D_2, O_2 and A_2 lie in one inertia line and also D', O_2 and A' lie in one inertia line.

Now, since $$(O_2, B_2) \{\equiv\} (O_2, B'),$$
we must have $$(O_2, A_2) (\equiv) (O_2, A'),$$
and therefore also in this case

$$(O_2, D_2) (\equiv) (O_2, D').$$

Combining this with the relation

$$(O_1, D_1) \mid \equiv \mid (O_2, D')$$

it follows by definition that:

$$(O_1, D_1) \equiv (O_2, D_2).$$

Thus the second part of the theorem is proved.

From Theorems 178 and 179 it follows that: if (O_1, D_1) and (O_2, D_2) be inertia pairs while (O_1, B_1) and (O_2, B_2) are separation pairs such that (O_1, B_1) is a before-conjugate to (O_1, D_1) and (O_2, B_2) is an after-conjugate to (O_2, D_2), then:

(1) If $$(O_1, D_1) \equiv (D_2, O_2),$$
we shall also have $$(O_1, B_1) \equiv (O_2, B_2).$$
(2) If $$(O_1, B_1) \equiv (O_2, B_2),$$
we shall also have $$(O_1, D_1) \equiv (D_2, O_2).$$

THEOREM 180

If (A_1, B_1), (A_2, B_2), (B_1, C_1), (B_2, C_2) *be pairs such that:*

$$(A_1, B_1) \mid\equiv\mid (A_2, B_2)$$

and

$$(B_1, C_1) \mid\equiv\mid (B_2, C_2),$$

and if C_1 *be distinct from* A_1, *then we shall also have*

$$(A_1, C_1) \mid\equiv\mid (A_2, C_2).$$

The elements A_1, B_1 and C_1 must lie in at least one general plane, say P_1, and since $A_2 B_2$ must either be parallel to $A_1 B_1$ or identical with it, while $B_2 C_2$ must either be parallel to $B_1 C_1$ or identical with it, it follows that there is a general plane, say P_2, either parallel to P_1 or identical with it which contains the elements A_2, B_2 and C_2.

Let P' be a general plane parallel to P_1 and P_2, and therefore distinct from both, and let (A', B') and (B', C') be pairs in P' such that:

$$(A_1, B_1) \square (A', B')$$

and

$$(B_1, C_1) \square (B', C').$$

Then, by Theorem 126, since $A_1 A'$ and $C_1 C'$ cannot lie in the same general line (owing to C_1 being distinct from A_1 and both of them lying in P_1), it follows that:

$$(A_1, C_1) \square (A', C').$$

Thus C' must be distinct from A'.

But now, since A_2, B_2 and C_2 lie in P_2 while A', B' and C' lie in the parallel general plane P', it follows that $A_2 B_2$ cannot be identical with $A'B'$, and $B_2 C_2$ cannot be identical with $B'C'$.

Thus, since

$$(A_1, B_1) \mid\equiv\mid (A_2, B_2)$$

and

$$(B_1, C_1) \mid\equiv\mid (B_2, C_2),$$

it follows that we must have

$$(A_2, B_2) \square (A', B')$$

and

$$(B_2, C_2) \square (B', C').$$

Thus, since C' is distinct from A', it follows that:

$$(A_2, C_2) \square (A', C').$$

But we have seen that:

$$(A_1, C_1) \square (A', C'),$$

and so

$$(A_1, C_1) \mid\equiv\mid (A_2, C_2).$$

Thus the theorem is proved.

Remarks

One special case of this theorem deserves attention.

If B_1 be linearly between A_1 and C_1, it follows, by Theorems 72, 91 and 117, that B' must be linearly between A' and C' and similarly B_2 must be linearly between A_2 and C_2.

We shall require this result in proving the next theorem.

Again, since the only congruence of optical pairs is co-directional, we may state the following result:

If (A_1, B_1), (A_2, B_2), (B_1, C_1), (B_2, C_2) be optical pairs such that:

$$(A_1, B_1) \equiv (A_2, B_2)$$

and

$$(B_1, C_1) \equiv (B_2, C_2),$$

then if B_1 be linearly between A_1 and C_1 we shall have B_2 linearly between A_2 and C_2 and also have

$$(A_1, C_1) \equiv (A_2, C_2).$$

Theorem 181

If (A_1, B_1), (A_2, B_2), (B_1, C_1), (B_2, C_2) be inertia pairs such that:

$$(A_1, B_1) \equiv (A_2, B_2)$$

and

$$(B_1, C_1) \equiv (B_2, C_2),$$

then if B_1 be linearly between A_1 and C_1, while B_2 is linearly between A_2 and C_2, we shall also have

$$(A_1, C_1) \equiv (A_2, C_2).$$

If a be an inertia line passing through A_2 and co-directional with A_1B_1 we may take a separation line b which passes through A_2 and is normal to both a and A_2B_2.

Let D_2 be an element in b such that (A_2, D_2) is conjugate to (A_2, B_2) and let (A_1, D_1) be a separation pair such that:

$$(A_1, D_1) \mid\equiv\mid (A_2, D_2).$$

Now A_1D_1 is co-directional with A_2D_2 while A_1B_1 is co-directional with a and so A_1D_1 must be normal to A_1B_1.

Since A_1B_1 and A_2B_2 are inertia lines, it follows that A_1B_1 and A_1D_1 lie in an inertia plane and also A_2B_2 and A_2D_2 lie in an inertia plane.

Then, since (A_2, D_2) is conjugate to (A_2, B_2), it follows that D_2B_2 is an optical line.

Let B_1' be an element in A_1B_1 such that (A_1, B_1') is an after- or before-conjugate to (A_1, D_1) according as (A_2, B_2) is an after- or before-conjugate to (A_2, D_2).

Then, by Theorem 179, since

$$(A_2, D_2) \equiv (A_1, D_1),$$

we must have $\quad (A_2, B_2) \equiv (A_1, B_1').$

But $\quad (A_1, B_1) \equiv (A_2, B_2)$

and so $\quad (A_1, B_1) \equiv (A_1, B_1').$

Thus, since $A_1 B_1$ is an inertia line we must have B_1' identical with B_1 and so, since (A_1, B_1) is conjugate to (A_1, D_1), it follows that $D_1 B_1$ is an optical line.

Now let the optical line through C_1 parallel to $B_1 D_1$ intersect $A_1 D_1$ in F_1 and let the separation line through B_1 parallel to $A_1 F_1$ intersect $C_1 F_1$ in E_1.

Then, since B_1 is linearly between A_1 and C_1, it follows, by Theorem 72, that D_1 is linearly between A_1 and F_1.

Let (D_2, F_2) be a pair such that:

$$(D_1, F_1) \mid\equiv\mid (D_2, F_2).$$

Then, by the remarks at the end of Theorem 180, D_2 will be linearly between A_2 and F_2 and we shall also have

$$(A_1, F_1) \mid\equiv\mid (A_2, F_2).$$

Now let the optical line through F_2 parallel to $D_2 B_2$ intersect $A_2 B_2$ in C_2', and let the separation line through B_2 parallel to $A_2 F_2$ intersect $F_2 C_2'$ in E_2.

Then we have $\quad (B_1, E_1) \mid\equiv\mid (D_1, F_1)$

and $\quad (D_1, F_1) \mid\equiv\mid (D_2, F_2),$

and therefore $\quad (B_1, E_1) \mid\equiv\mid (D_2, F_2).$

But we have also $\quad (D_2, F_2) \mid\equiv\mid (B_2, E_2),$

and so $\quad (B_1, E_1) \mid\equiv\mid (B_2, E_2).$

But now, since $B_1 E_1$ is parallel to $A_1 D_1$, it must be normal to $B_1 C_1$, and since $E_1 C_1$ is an optical line, it follows that (B_1, E_1) is conjugate to (B_1, C_1).

Similarly, since $B_2 E_2$ is parallel to $A_2 D_2$, it must be normal to $B_2 C_2'$ and, since $E_2 C_2'$ is an optical line, it follows that (B_2, E_2) is conjugate to (B_2, C_2').

But now, since D_2 is linearly between A_2 and F_2, it follows that B_2 is linearly between A_2 and C_2'.

If then B_2 be *after* A_2 we must have C_2' *after* B_2, while if B_2 be *before* A_2 we must have C_2' *before* B_2.

But, since B_2 is linearly between A_2 and C_2, it follows that if B_2 be

after A_2 we must have C_2 *after* B_2, while if B_2 be *before* A_2 we must have C_2 *before* B_2.

Thus C_2' is *after* or *before* B_2 according as C_2 is *after* or *before* B_2.

But, since (B_1, C_1) and (B_2, C_2) are inertia pairs such that:

$$(B_1, C_1) \equiv (B_2, C_2),$$

it follows that C_2 is *after* or *before* B_2 according as C_1 is *after* or *before* B_1 and therefore C_2' is *after* or *before* B_2 according as C_1 is *after* or *before* B_1.

Thus, since $$(B_1, E_1) \equiv (B_2, E_2),$$

it follows, by Theorem 179, that:

$$(B_1, C_1) \equiv (B_2, C_2'),$$

and since $$(B_1, C_1) \equiv (B_2, C_2),$$

it follows that: $$(B_2, C_2) \equiv (B_2, C_2').$$

Thus, since these pairs lie in the same inertia line, we must have C_2' identical with C_2.

But now C_2 will be *after* or *before* A_2 according as C_1 is *after* or *before* A_1 and so (A_2, C_2) will be an after- or before-conjugate to (A_2, F_2) according as (A_1, C_1) is an after- or before-conjugate to (A_1, F_1).

Thus, since $$(A_1, F_1) \equiv (A_2, F_2),$$

it follows, by Theorem 179, that:

$$(A_1, C_1) \equiv (A_2, C_2),$$

and so the theorem is proved.

Theorem 182

If (A_1, B_1), (A_2, B_2), (B_1, C_1), (B_2, C_2) be separation pairs such that:

$$(A_1, B_1) \equiv (A_2, B_2)$$

and $$(B_1, C_1) \equiv (B_2, C_2),$$

then if B_1 be linearly between A_1 and C_1 while B_2 is linearly between A_2 and C_2 we shall also have

$$(A_1, C_1) \equiv (A_2, C_2).$$

Let (A_1, D_1) and (A_2, D_2) be inertia pairs which are after-conjugates to (A_1, B_1) and (A_2, B_2) respectively.

Then, since A_1D_1 and A_2D_2 are inertia lines, it follows that A_1D_1 and A_1B_1 lie in an inertia plane and A_2D_2 and A_2B_2 lie in an inertia plane.

Since (A_1, D_1) is conjugate to (A_1, B_1), it follows that B_1D_1 is an optical line, and similarly B_2D_2 is an optical line.

Now let the optical line through C_1 parallel to B_1D_1 intersect A_1D_1

in F_1, and let the optical line through C_2 parallel to B_2D_2 intersect A_2D_2 in F_2.

Then, since B_1 is linearly between A_1 and C_1, it follows, by Theorem 72, that D_1 is linearly between A_1 and F_1.

Similarly D_2 is linearly between A_2 and F_2.

Let the inertia line through B_1 parallel to A_1F_1 intersect C_1F_1 in E_1, and let the inertia line through B_2 parallel to A_2F_2 intersect C_2F_2 in E_2.

Then, since (A_1, D_1) is an after-conjugate to (A_1, B_1), we must have D_1 *after* A_1 and, since D_1 is linearly between A_1 and F_1, we must have F_1 *after* D_1.

Thus we must have E_1 *after* B_1 and in a similar manner we can show that E_2 must be *after* B_2.

But now, since (A_1, D_1) is conjugate to (A_1, B_1), it follows that A_1D_1 is normal to A_1B_1, and since B_1E_1 is parallel to A_1D_1 while A_1, B_1 and C_1 lie in one general line, it follows that B_1E_1 is normal to B_1C_1.

Thus, since C_1E_1 (that is C_1F_1) is an optical line, it follows that (B_1, E_1) is conjugate to (B_1, C_1) and (A_1, F_1) is conjugate to (A_1, C_1).

Further, D_1 is *after* A_1 and F_1 is *after* D_1 and so F_1 is *after* A_1.

Thus (B_1, E_1) and (A_1, F_1) are after-conjugates to (B_1, C_1) and (A_1, C_1) respectively.

Similarly (B_2, E_2) and (A_2, F_2) are after-conjugates to (B_2, C_2) and (A_2, C_2) respectively.

But now, since $\qquad (A_1, B_1) \equiv (A_2, B_2),$

while (A_1, D_1) and (A_2, D_2) are after-conjugates to (A_1, B_1) and (A_2, B_2) respectively, it follows, by Theorem 179, that:

$$(A_1, D_1) \equiv (A_2, D_2).$$

Similarly, since $\qquad (B_1, C_1) \equiv (B_2, C_2),$

while (B_1, E_1) and (B_2, E_2) are after-conjugates to (B_1, C_1) and (B_2, C_2) respectively, it follows that:

$$(B_1, E_1) \equiv (B_2, E_2).$$

But we clearly have $\qquad (B_1, E_1) \mid\equiv\mid (D_1, F_1)$

and $\qquad (B_2, E_2) \mid\equiv\mid (D_2, F_2).$

Thus we have $\qquad (D_1, F_1) \equiv (D_2, F_2).$

But, since D_1 is linearly between A_1 and F_1, while D_2 is linearly between A_2 and F_2, it follows, by Theorem 181, that:

$$(A_1, F_1) \equiv (A_2, F_2).$$

We have however seen that (A_1, F_1) and (A_2, F_2) are after-conjugates to (A_1, C_1) and (A_2, C_2) respectively, and so it follows, by Theorem 179, that

$$(A_1, C_1) \equiv (A_2, C_2).$$

Thus the theorem is proved.

Theorem 183

If A and B be two distinct elements and E be any element in AB distinct from A and B, while F is an element in AB such that:

$$(A, E) \,|\!\equiv\!|\, (F, B),$$

then we shall have $$(A, F) \,|\!\equiv\!|\, (E, B).$$

Let a be a general line parallel to AB and let a general line through A intersect a in A' while parallel general lines through B and E intersect a in B' and E' respectively.

Finally let a general line through F parallel to AA' or BB' intersect a in F'.

Now we clearly have

$$(E, E') \,\square\, (B, B'),$$

and so $$(E, E') \,|\!\equiv\!|\, (B, B').$$

But, since E' and A lie in parallel general lines, they must be distinct, and so, by Theorem 180,

$$(A, E') \,|\!\equiv\!|\, (F, B').$$

Now F cannot coincide with A, for then E would require to coincide with B, contrary to hypothesis, and so FB' must be parallel to AE'. Thus we must have

$$(A, F) \,\square\, (E', B').$$

But we obviously have

$$(E, B) \,\square\, (E', B')$$

and so $$(A, F) \,|\!\equiv\!|\, (E, B).$$

Thus the theorem is proved.

Theorem 184

If A_0, A_1 and C be three distinct elements such that A_1 is linearly between A_0 and C and if $A_2, A_3, A_4, \ldots$ be elements such that:

A_1 is linearly between A_0 and A_2,
A_2 is linearly between A_0 and A_3,
....
....

and such that: $$(A_0, A_1) \equiv (A_1, A_2) \equiv (A_2, A_3) \ldots,$$

then there are not more than a finite number of the elements $A_1, A_2, A_3, \ldots$ *linearly between* A_0 *and* C.

It is evident that all the series of elements $A_1, A_2, A_3, \ldots$ lie in the general line $A_0 C$ which we shall call a.

We shall first prove the theorem for the case where a is an inertia line and C is *after* A_0.

We shall suppose that a lies in an inertia plane P.

Now since A_1 is linearly between A_0 and C we must have A_1 *after* A_0, and so if we take two generators of P of opposite sets passing through A_0 and A_1 respectively, they will intersect in some element, say B_0, which will be *after* A_0 and *before* A_1 and must lie outside a.

Let b be an inertia line passing through B_0 and parallel to a and let optical lines parallel to $A_0 B_0$ and passing through $A_1, A_2, A_3, \ldots$ intersect b in the elements $B_1, B_2, B_3, \ldots$ respectively.

Now, since A_1 is *after* A_0 and, since further:

A_1 is linearly between A_0 and A_2,
A_2 is linearly between A_0 and A_3,
....
....

it follows that:

A_1 is *after* A_0; A_2 is *after* A_1; A_3 is *after* A_2; ....

Thus, since $(A_0, A_1) \equiv (A_1, A_2) \equiv (A_2, A_3) \ldots$,

it follows that: A_1 is the mean of A_0 and A_2,
A_2 is the mean of A_1 and A_3,
....
....

But now, by construction, we have

$$(A_0, A_2) \square (B_0, B_2),$$

and so, since A_1 is the mean of A_0 and A_2, and since $A_1 B_1$ is parallel to $A_0 B_0$ and $A_2 B_2$, it follows that B_1 is the mean of B_0 and B_2 and so, by Theorem 80, $A_2 B_1$ is parallel to $A_1 B_0$.

Similarly $A_3 B_2$ is parallel to $A_2 B_1$ and so on.

Thus, since $A_1 B_0$ is an optical line, it follows that $A_2 B_1, A_3 B_2, \ldots$ are all optical lines and so $A_1, A_2, A_3, \ldots$ mark steps taken along a with respect to b.

But, since a and b do not intersect, it follows, by Post. XVII, that C may be surpassed in a finite number of steps taken from A_0.

Thus there cannot be more than a finite number of the elements $A_1, A_2, A_3, \ldots$ linearly between A_0 and C.

Similarly if C be *before* A_0 the same result follows by using the (b) form of Post. XVII, and so the theorem is proved for the case where a is an inertia line.

Consider next the case where a is a separation line and let b be any inertia line which passes through A_0.

Then a and b determine an inertia plane which we shall call P.

Now one of the generators of P which pass through A_1 intersects b in an element which lies in the α sub-set of A_1, while the other generator intersects b in an element which lies in the β sub-set of A_1.

Let the former of these generators be called f_1, and let it intersect b in the element A_1'.

Then, since A_1 does not lie in b, it follows that A_1' is *after* A_1 and so, since A_0A_1 is a separation line, we must also have A_1' *after* A_0.

Let $f_0, f_2, f_3, f_4, \ldots$ and f_c be generators of P parallel to f_1 and passing through A_0, A_2, A_3, A_4, ... and C respectively.

Further, let $f_2, f_3, f_4, \ldots$ and f_c intersect b in A_2', A_3' A_4', ... and C' respectively.

Then, since A_1' is *after* A_0, it follows that f_1 is an after-parallel of f_0, and since A_1 is linearly between A_0 and C, it follows that f_c is an after-parallel of f_1 and so C' is *after* A_1'.

Further, A_1 is linearly between A_0 and A_2,
A_2 is linearly between A_0 and A_3,
....
....

Thus we have f_1 is an after-parallel of f_0,
f_2 is an after-parallel of f_1,
f_3 is an after-parallel of f_2,
....
....

Thus we have
A_1 is linearly between A_0 and A_2,
A_2 is linearly between A_1 and A_3,
A_3 is linearly between A_2 and A_4,
....
....

But, since $(A_0, A_1) \equiv (A_1, A_2) \equiv (A_2, A_3) \ldots$,
it follows that: A_1 is the mean of A_0 and A_2,
A_2 is the mean of A_1 and A_3,
....
....

Thus, by Theorem 81,

$$A_1' \text{ is the mean of } A_0 \text{ and } A_2',$$
$$A_2' \text{ is the mean of } A_1' \text{ and } A_3',$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

and so $$(A_0, A_1') \equiv (A_1', A_2') \equiv (A_2', A_3') \ldots .$$

Thus, by the first case of the theorem, there cannot be more than a finite number of the elements A_1', A_2', A_3' linearly between A_0 and C'.

But each of these elements which is linearly between A_0 and C' corresponds to one of the series $A_1, A_2, A_3, \ldots$ which is linearly between A_0 and C, while any one which is not linearly between A_0 and C' corresponds to one of the series $A_1, A_2, A_3, \ldots$ which is not linearly between A_0 and C.

Thus there are not more than a finite number of the elements A_1, $A_2, A_3, \ldots$ linearly between A_0 and C, and so the theorem holds when a is a separation line.

As regards the case where a is an optical line and C is *after* A_0 we may proceed just as we have done for the case where a is a separation line.

In this case a is one of the generators of the inertia plane P, while $f_0, f_1, f_2, \ldots f_c$ will be generators of the opposite set.

The result then follows in a similar manner.

In the case where a is an optical line and C is *before* A_0 we also make use of a similar method except that the element C' in the inertia line b will be *before* A_0 instead of *after* it.

Thus the theorem holds in all cases.

Remarks

It will be observed that the above theorem is equivalent to the *Axiom of Archimedes* and has been deduced by the help of Post. XVII.

In our remarks on the introduction of this postulate, its analogy to the Axiom of Archimedes was pointed out together with the fact that the postulate contains no reference to *congruence*.

Having defined congruence of pairs we are able to deduce the Axiom of Archimedes in the usual form as given above.

Definitions. If A and B be two distinct elements, then the set of all elements lying linearly between A and B will be called the *segment* AB.

The elements A and B will be called the *ends* of the segment, but are not included in it.

The set of elements obtained by including the ends will be called a *linear interval.*

Since an element which is linearly between two elements A and B is linearly between B and A, it follows, from the definition, that the segment BA is the same as the segment AB, even in those cases where one of the two elements: A, B is *after* the other.

The same remark applies to linear intervals.

If A and B be two distinct elements, then the set of elements such as X where B is linearly between A and X may be called the *prolongation of the segment AB beyond B.*

Such a set of elements will also be spoken of as a *general half-line.*

The element B will be called the *end* of the general half-line.

We shall describe segments, linear intervals and general half-lines as *optical, inertia,* or *separation,* according as they lie in optical, inertia, or separation lines.

It is easy to see that any element B in a general line a divides the remaining elements of the general line into two sets such that B is linearly between any two elements of opposite sets, but is not linearly between any two elements of the same set.

For let P be any inertia plane which contains a and let b be a generator of P which passes through B, and which, in case a is an optical line, we shall suppose to be distinct from a.

Then through every element of a which is distinct from B there will pass an optical line which is parallel to b.

Those elements of a which lie in optical lines which are after-parallels of b constitute the one set and those which lie in optical lines which are before-parallels of b constitute the other set.

It is then obvious, from the definition of linearly between, that B is linearly between any two elements of a belonging to opposite sets, but is not linearly between any two elements of a belonging to the same set.

It is clear that these sets are general half-lines.

If elements X and Y lie in the same general half-line whose end is B, they will be said to lie *on the same side* of B.

If, on the other hand, B be linearly between X and Y, then these elements will be said to lie *on opposite sides* of B.

A general half-line whose end is B and which contains an element X may be denoted by the *general half-line BX.*

It is also easy to see that any general line b in a general plane P divides the remaining elements of P into two sets such that if A and C be any two elements of opposite sets then b will intersect AC in an element linearly between A and C; while if A and A' be two elements

of the same set, then b will not intersect AA' in any element linearly between A and A'.

For consider the elements of P exclusive of those lying in b and, when dealing with them, let us use the expression: "A is opposite to C" as an abbreviation for: "b intersects AC in an element linearly between A and C".

Let B be such an element of intersection and let A' be a second element which is opposite to C.

If A' should happen to lie in AC then, from what we have just shown, it follows that A' is not opposite to A.

If next we take the case where A' does not lie in AC then, since B is linearly between A and C and, since neither A, C nor A' lie in b, it follows, by Theorem 127 (2), that A' is not opposite to A. Thus (i) *in all cases, if A and A' are both opposite to C, then A' is not opposite to A.*

Next suppose that A is opposite to C while A_1 is another element of P which is not opposite to A and does not lie in b.

If A_1 should happen to lie in AC then, from the linear analogue already considered it follows that A_1 is opposite to C.

If A_1 does not lie in AC then, since B is linearly between A and C and since neither A, C nor A_1 lie in b, it follows, by Theorem 127 (1), that A_1 is opposite to C. Thus (ii) *in all cases, if A is opposite to C while A_1 is not opposite to A, we must have A_1 opposite to C.*

If A_2 be a second element which is not opposite to A, it follows by (ii) that A_2 is opposite to C.

But, since both A_1 and A_2 are opposite to C, it follows by (i) that A_2 is not opposite to A_1. Thus all the elements which are not opposite to A are opposite to C and no one of them is opposite to any other of them.

Similarly, all the elements which are not opposite to C are opposite to A, A_1, A_2, etc. and no one of them is opposite to any other of them.

This shows that the general plane P is divided by the general line b in the manner above stated.

If elements X and Y lie in the general plane P, but not in the general line b, they will be said to lie *on the same side* of b if they both lie in the same set and will be said to lie *on opposite sides* of b if X lies in one of the sets and Y in the other set.

Definition. If a general line b lies in a general plane P, then either of the sets of elements on one side of b will be called a *general half-plane.*

The general line b will be called the *boundary* of the general half-plane.

The following important result which may be conveniently expressed in the nomenclature of general half-lines can be easily proved.

If (A_1, B_1), (A_2, B_2), (A_1, C_1), (A_2, C_2) be inertia, optical or separation pairs such that:

$$(A_1, B_1) \equiv (A_2, B_2),$$

and $$(A_1, C_1) \equiv (A_2, C_2),$$

then if B_1 be linearly between A_1 and C_1 and if C_2 lies in the general half-line $A_2 B_2$, we shall also have B_2 linearly between A_2 and C_2.

In the case of optical pairs the above congruences imply that $A_1 B_1$ and $A_2 B_2$ are the same or parallel optical lines, but nothing of this sort is implied in the case of inertia or separation pairs.

In all cases there is an element, say C_2', in $A_2 B_2$ and on the opposite side of B_2 to that on which A_2 lies and such that

$$(B_1, C_1) \equiv (B_2, C_2').$$

Then in all cases it follows that:

$$(A_1, C_1) \equiv (A_2, C_2'),$$

and so $$(A_2, C_2) \equiv (A_2, C_2').$$

But C_2 and C_2' both lie in $A_2 B_2$ and on the same side of A_2, and must therefore be identical.

Thus B_2 is linearly between A_2 and C_2, and

$$(B_1, C_1) \equiv (B_2, C_2).$$

Definitions. If (A, B) and (C, D) be inertia or optical pairs in which B is *after* A and D *after* C, or if (A, B) and (C, D) be separation pairs, then, in respect of magnitude:

(1) If $(A, B) \equiv (C, D)$ we shall say that the segment AB is *equal to* the segment CD.

(2) If $(A, B) \equiv (C, E)$ where E is any element linearly between C and D, we shall say that the segment AB is *less than* the segment CD.

(3) If $(A, B) \equiv (C, F)$ where F is any element such that D is linearly between C and F, we shall say that the segment AB is *greater than* the segment CD.

In the case of separation or inertia segments we must always have either:

AB is equal to CD,

or AB is less than CD,

or AB is greater than CD.

In the case of optical segments, however, this is only true provided they lie in co-directional optical lines.

Again, if (A, B) and (C, D) be inertia or optical pairs in which B is *after* A and D *after* C, or if they be separation pairs, and if E, F, G be elements such that F is linearly between E and G while

$$(A, B) \equiv (E, F)$$

and
$$(C, D) \equiv (F, G),$$

we shall say that *the segment EG is equal to the sum of the segments AB and CD in respect of magnitude.*

It is evident that two optical segments can only have a sum in this sense provided they lie in co-directional optical lines, whereas two inertia segments or two separation segments always have a sum.

In the above definitions the words *linear interval* may be substituted for the word *segment.*

If A and B be any two distinct elements and if C be any element such that B is linearly between A and C and if a and b be taken to denote the segments AB and BC respectively, then we may express the result of Theorem 183 in the form:

$$a + b = b + a.$$

If D be any other element such that C is linearly between B and D, and if we denote the segment CD by c; then, by application of Theorem 183 alternately to a pair of segments lying towards one end of the total interval AD and then to a pair of segments lying towards the other end, we obtain successively:

$$a + b + c = b + a + c,$$
$$b + a + c = b + c + a,$$
$$b + c + a = c + b + a,$$
$$c + b + a = c + a + b,$$
$$c + a + b = a + c + b,$$
$$a + c + b = a + b + c.$$

It thus appears that we may regard both the Commutative Law and the Associative Law as holding for the addition of segments, or of linear intervals.

Having thus introduced the idea of a segment (or linear interval) being equal to the sum of two others we can obviously have any *multiple* and also (as follows from the remarks at the end of Theorem 81) any *sub-multiple* of a given segment (or linear interval): using the terms "multiple" and "sub-multiple" in the ordinary sense.

We may also clearly have a segment (or linear interval) equal to any proper or improper fractional part of a given one.

The criterion of *proportion* given by Euclid, and which is probably due to Eudoxos, is clearly applicable in our geometry.

This criterion is as follows:

If we have four magnitudes of which the first and second are *of one kind** and the third and fourth also *of one kind* and if, for all values of m and n (where m and n are integers), we have m (first magnitude) is greater than, equal to, or less than n (second magnitude) according as m (third magnitude) is greater than, equal to, or less than n (fourth magnitude), then the four magnitudes are *in proportion* in such a way that the first magnitude is to the second as the third is to the fourth.

We shall also express the proportion of the four magnitudes by saying that the *ratio* of the first to the second is equal to the *ratio* of the third to the fourth.

This might be regarded merely as another form of words expressing the same fact, without assigning any specific definition to the term *ratio* taken by itself; since, in all cases where the term is used, it is possible to get rid of it by means of a circumlocution. As, however, it is desirable not to use a technical term without definition, we may define *ratio*, when employed in the Euclidean sense, thus:

Definition. The *ratio* of two magnitudes of one kind is the mode of distribution of the multiples m of the one magnitude among the multiples n of the other in respect of the relations of *greater than*, *equal to*, and *less than* for all integral values of m and n.

Later on, for purposes of manipulation, we shall find it convenient to represent ratios by what are called *real numbers* and to associate positive and negative signs with them, but, in the mean time this is unnecessary.

If, in a proportion, for some particular values of m and n (say m_1 and n_1) we should have m_1 (first magnitude) is equal to n_1 (second magnitude) and, along with that, m_1 (third magnitude) is equal to n_1 (fourth magnitude), then the first and second magnitudes are commensurable as are also the third and fourth, and their common ratio is that of n_1 to m_1: written $n_1 : m_1$.

This, however, is by no means always so and, when it is not the case,

* On this point see M. J. M. Hill's *Theory of Proportion*, Article 3.

It is very important to observe the sense in which the words "of one kind" are employed in the above criterion.

Thus, *segments of optical lines can only be deemed magnitudes of one kind provided that the optical lines are co-directional.*

the magnitudes are said to stand in an incommensurable ratio to one another.

The criterion then reduces to the form that, for all integral values of m and n we have m (first magnitude) is greater than or less than n (second magnitude) according as m (third magnitude) is greater than or less than n (fourth magnitude).

It has been shown by Stolz that this is a sufficient criterion even when the magnitudes stand in a commensurable ratio.

The abstract theory of proportion, as treated by Euclid in his fifth book (or by the late Professor M. J. M. Hill in his work on the subject), will be assumed in what follows and we shall concern ourselves only with the application of it to our geometry.

Certain results regarding the proportion of segments may easily be shown to hold for all types of general line, by using methods such as are employed in the following theorem.

THEOREM 185

If O, A_1 and B_1 be three distinct elements not lying in one general line, while C_1 is any other element lying in the general half-line OA_1, and if a general line through C_1 parallel to A_1B_1 intersects OB_1 in an element D_1, then

(1) *segment OA_1 : segment OC_1 = segment OB_1 : segment OD_1;*

(2) *segment OA_1 : segment OC_1 = segment A_1B_1 : segment C_1D_1.*

In the general half-line OA_1 let elements $A_2, A_3, \ldots A_m$ be taken such that:

A_1 is the mean of O and A_2,

A_2 is the mean of A_1 and A_3,

....

....

A_{m-1} is the mean of A_{m-2} and A_m,

and let general lines through $A_2, A_3, \ldots A_m$ be taken parallel to A_1B_1 and meeting OB_1 in the elements $B_2, B_3, \ldots B_m$ respectively. Then we know that:

B_1 is the mean of O and B_2,

B_2 is the mean of B_1 and B_3,

....

....

B_{m-1} is the mean of B_{m-2} and B_m.

Then segment $OA_m = m$ (segment OA_1)

and segment $OB_m = m$ (segment OB_1).

Similarly in the general half-line OC_1 (i.e. OA_1) let elements C_2, C_3, ... C_n be taken such that:

C_1 is the mean of O and C_2,
C_2 is the mean of C_1 and C_3,
....
....
C_{n-1} is the mean of C_{n-2} and C_n,

and let general lines through C_2, C_3, ... C_n be taken parallel to C_1D_1

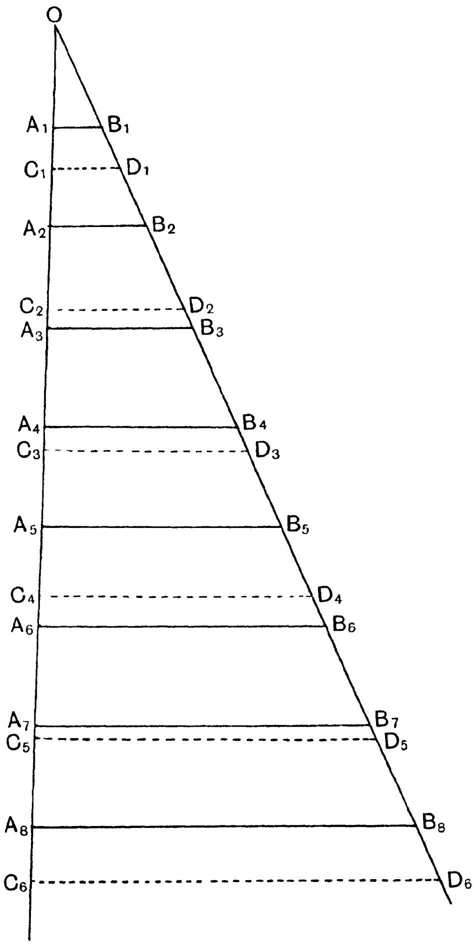


Fig. 46.

and meeting OC_1 in the elements D_2, D_3, ... D_n respectively. Then, as before, we get

$$\text{segment } OC_n = n \text{ (segment } OC_1\text{)},$$

and $$\text{segment } OD_n = n \text{ (segment } OD_1\text{)}.$$

But now we shall have

(i) C_n linearly between O and A_m,

or (ii) C_n identical with A_m,

or (iii) A_m linearly between O and C_n,

according respectively as we have

(i) D_n linearly between O and B_m,

or (ii) D_n identical with B_m,

or (iii) B_m linearly between O and D_n.

Thus segment OA_m is greater than, equal to, or less than segment OC_n according as segment OB_m is greater than, equal to, or less than segment OD_n; which is the criterion that

$$\text{segment } OA_1 : \text{segment } OC_1 = \text{segment } OB_1 : \text{segment } OD_1.$$

This proves the first part of the theorem.

In order to prove the second part of the theorem let the general line through D_1 parallel to OA_1 intersect A_1B_1 in the element E and let the general line through E parallel to OB_1 intersect OA_1 in the element F.

Then clearly we have

$$(C_1, D_1) \mid\equiv\mid (A_1, E)$$

and

$$(C_1, A_1) \mid\equiv\mid (O, F).$$

Thus, by Theorem 183,

$$(C_1, O) \mid\equiv\mid (A_1, F).$$

But, since EF is parallel to B_1O, we have, by the first part of the theorem,

$$\text{segment } A_1O : \text{segment } A_1F = \text{segment } A_1B_1 : \text{segment } A_1E.$$

Thus, since

$$\text{segment } A_1F = \text{segment } C_1O$$

and

$$\text{segment } A_1E = \text{segment } C_1D_1,$$

we have

$$\text{segment } A_1O : \text{segment } C_1O = \text{segment } A_1B_1 : \text{segment } C_1D_1.$$

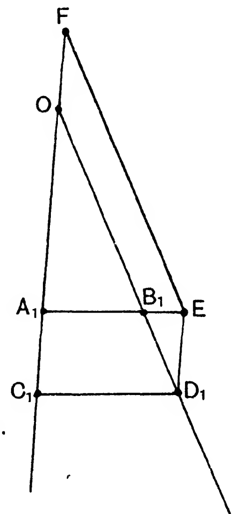


Fig. 47.

That is

$$\text{segment } OA_1 : \text{segment } OC_1 = \text{segment } A_1B_1 : \text{segment } C_1D_1,$$

as was to be proved.

Remarks

It should be noted that certain transformations of the above results may be deduced by the abstract theory of proportion in all cases; while certain other transformations are only permissible in those cases

where the four terms of the proportion are all segments of the same kind.

Thus if all four terms are either segments of separation lines, or else all segments of inertia lines, part (1) may be transformed into:

$$\text{segment } OA_1 : \text{segment } OB_1 = \text{segment } OC_1 : \text{segment } OD_1;$$

while part (2) may be transformed into:

$$\text{segment } A_1B_1 : \text{segment } OA_1 = \text{segment } C_1D_1 : \text{segment } OC_1;$$

but these would not be permissible in other cases.

Another important point to be observed is that if OA_1 should be a separation line and OB_1 an inertia line normal to OA_1 (or conversely), while A_1B_1 is an optical line, then C_1D_1 will also be an optical line and so it follows that: *separation segments are proportional to their conjugate inertia segments.*

THEOREM 186

If B and C be two distinct elements in a separation line and O be their mean, and if A be any element in a separation line a which passes through O and is normal to BC, then:

$$(A, B) \equiv (A, C).$$

Since a is normal to BC and since they are both separation lines, it follows that a and BC lie in a separation plane, say S.

If the element A should happen to coincide with O, then, since BC is a separation line, the theorem obviously holds.

Suppose next that A does not coincide with O and let d be an inertia line passing through A and normal to S.

Let P be the inertia plane containing a and d.

Now, since d is normal to S, it follows that BC is normal to d, and since BC is also normal to a, and since a and d intersect and lie in P, it follows that BC is normal to P.

Let D be the one single element common to d and the α sub-set of B.

Then BC and BD determine a general plane, say Q, which must be either an optical plane or an inertia plane, since BD is an optical line.

But now P and Q have the general line OD in common, and since BC is normal to P, it follows that BC is normal to OD.

If Q were an optical plane OD would require to be an optical line, while if Q were an inertia plane OD would be an inertia line.

But, since BD is an optical line in Q and since BD and OD intersect, it follows that Q cannot be an optical plane.

Thus Q must be an inertia plane and OD must be an inertia line normal to the separation line BC.

Thus, since O is the mean of B and C, it follows that B, C and D are three corners of an optical parallelogram of which O is the centre.

Thus CD is an optical line.

But, since AD is normal to S, it must be normal to both AB and AC.

Also, since D is in the α sub-set of B and is distinct from B, it follows that D is *after* B.

Thus, since AB is a separation line while AD is an inertia line, it follows that D is *after* A, and accordingly (A, D) is an after-conjugate to both (A, B) and (A, C).

Thus we have
$$(A, B) \equiv (A, C),$$
and so the theorem is proved.

Theorem 187

If A, B and C be three distinct elements which lie in a separation plane S, but do not all lie in one general line, and if O be the mean of B and C while
$$(A, B) \equiv (A, C),$$
then AO is normal to BC.

Let d be an inertia line passing through A and normal to S and let P be the inertia plane containing d and AO.

Then d is normal to both AB and AC and, since
$$(A, B) \equiv (A, C),$$
there is one definite element, say D, in d such that (A, D) is an after-conjugate to both (A, B) and (A, C).

Thus BD and CD are optical lines and, since they intersect, they must lie in an inertia plane, say Q.

But now, since O is the mean of B and C, it follows that B, C and D are three corners of an optical parallelogram whose centre is O, and therefore BC is normal to OD.

But OD is common to both Q and P, while BC (since it lies in S) is normal to AD, which also lies in P.

Thus BC is normal to two intersecting general lines in P and therefore BC is normal to P.

But AO lies in P and therefore AO is normal to BC.

Thus the theorem is proved.

Theorem 188

If A, B and C be three distinct elements which lie in a separation plane S, but do not all lie in one general line, and if

$$(A, B) \equiv (A, C),$$

and if O be an element in BC such that AO is normal to BC, then O is the mean of B and C.

Let O' be the mean of B and C.

Then, by Theorem 187, AO' is normal to BC, and, by hypothesis, AO is normal to BC.

But both AO' and AO pass through the element A and lie in the separation plane S and we have already seen that there is only one general line in a given separation plane which passes through a given element and is normal to another general line in the separation plane.

Thus AO must be identical with AO' and therefore O must be identical with O'.

It follows that O is the mean of B and C and so the theorem is proved.

Definition. If A, B, C be three distinct elements which do not all lie in one general line, then the three segments AB, BC, CA, together with the three elements A, B, C, will be called a *general triangle*, or briefly a *triangle* in an inertia optical, or separation plane, as the case may be.

The elements A, B, C will be called the *corners* while the segments AB, BC, CA will be called the *sides* of the general triangle.

Theorem 189

If A_1, B_1, C_1 be the corners of a triangle in a separation plane P_1 and A_2, B_2, C_2 be the corners of a triangle in a separation plane P_2 and if further

$$(C_1, A_1) \equiv (C_2, A_2),$$
$$(C_1, B_1) \equiv (C_2, B_2),$$

while B_1C_1 is normal to A_1C_1, and B_2C_2 is normal to A_2C_2, then we shall also have

$$(A_1, B_1) \equiv (A_2, B_2).$$

In order to prove this theorem we shall consider a number of special cases on which the general proof is made to depend.

Case I. *B_2 identical with B_1 and C_2 identical with C_1, while P_2 is identical with P_1.*

In this case, since the separation lines A_1C_1 and A_2C_1 are both normal

to B_1C_1 and both lie in the separation plane P_1 and pass through the element C_1, they must be identical.

If further A_2 should coincide with A_1 the result is obvious, and so we shall suppose that A_2 does not coincide with A_1.

Now, since (C_1, A_1), etc. lie in the separation plane P_1, they must all be separation pairs and, since in this case

$$(C_1, A_2) \equiv (A_2, C_1),$$

it follows that:
$$(A_2, C_1) \equiv (C_1, A_1).$$

Thus C_1 must be the mean of A_1 and A_2 and therefore, by Theorem 186, we have

$$(B_1, A_1) \equiv (B_1, A_2),$$

or
$$(A_1, B_1) \equiv (A_2, B_2).$$

CASE II. *B_2 identical with B_1 and C_2 identical with C_1, while P_1 and P_2 lie in the same separation threefold W.*

If P_2 should be identical with P_1 this case reduces to Case I, and so we shall suppose them distinct.

Now, since A_1, B_1 and A_2 are three distinct elements in W which do not lie in one general line, it follows that A_1, B_1 and A_2 lie in a separation plane, say R, which must be distinct from both P_1 and P_2, since these latter two separation planes are supposed distinct.

Similarly A_1, C_1 and A_2 lie in a separation plane, say S, which is also distinct from P_1 and P_2.

Now let O be the mean of A_1 and A_2.

Then, by Theorem 187, since

$$(C_1, A_1) \equiv (C_1, A_2),$$

it follows that C_1O is normal to A_1A_2.

But, since B_1C_1 is normal to A_1C_1 and to A_2C_1 which are distinct intersecting separation lines, it follows that B_1C_1 is normal to S and therefore must be normal to A_1A_2.

Thus A_1A_2 is normal to the two intersecting separation lines B_1C_1 and C_1O and must therefore be normal to every general line in the general plane containing them.

It follows that A_1A_2 is normal to B_1O.

But now, by Theorem 186, since O is the mean of A_1 and A_2 and, since B_1, A_1, A_2 lie in a separation plane, it follows that:

$$(B_1, A_1) \equiv (B_1, A_2),$$

or
$$(A_1, B_1) \equiv (A_2, B_2).$$

CASE III. *C_2 identical with C_1 and P_2 identical with P_1.*

Let b be a separation line passing through C_1 and normal to P_1 and let B' be an element in b such that:

$$(C_1, B') \equiv (C_1, B_1).$$

Then we shall also have

$$(C_1, B') \equiv (C_1, B_2).$$

Now the separation plane P and the separation line b determine a separation threefold, say W, which contains $A_1, B_1, C_1, A_2, B_2, B'$.

Again, since $B'C_1$ is normal to P, it must be normal to C_1A_1, C_1A_2, C_1B_1 and C_1B_2.

Then since $$(C_1, B_1) \equiv (C_1, B'),$$
it follows, by Case II, that:
$$(A_1, B_1) \equiv (A_1, B') \qquad \ldots\ldots(1).$$
Again since $$(C_1, A_1) \equiv (C_1, A_2),$$
it follows, by Case II, that:
$$(A_1, B') \equiv (A_2, B') \qquad \ldots\ldots(2).$$
Further, by Case II, since
$$(C_1, B') \equiv (C_1, B_2),$$
it follows that: $$(A_2, B') \equiv (A_2, B_2) \qquad \ldots\ldots(3).$$
Thus, from (1), (2) and (3), it follows that:
$$(A_1, B_1) \equiv (A_2, B_2).$$

CASE IV. *P_2 either identical with P_1 or parallel to P_1.*

There is, as we have already seen, one single element, say A', such that:

$$(C_1, A_1) \,|\!\equiv\!|\, (C_2, A').$$

Similarly there is one single element, say B', such that:

$$(C_1, B_1) \,|\!\equiv\!|\, (C_2, B').$$

Now, since P_2 is either identical with P_1 or parallel to P_1, it follows that P_2 must contain C_2A' and C_2B'.

Also, since C_2A' must be either parallel to C_1A_1 or identical with it, and since C_2B' must be either parallel to C_1B_1 or identical with it, then since B_1C_1 is normal to A_1C_1, it follows that $B'C_2$ is normal to $A'C_2$.

But now, by Theorem 180, we must have

$$(A_1, B_1) \,|\!\equiv\!|\, (A', B').$$

Also since $$(C_1, A_1) \equiv (C_2, A_2),$$
it follows that: $$(C_2, A_2) \equiv (C_2, A'),$$

and, since $(C_1, B_1) \equiv (C_2, B_2)$,

it follows that: $(C_2, B_2) \equiv (C_2, B')$.

Thus, by Case III, it follows that:

$$(A', B') \equiv (A_2, B_2).$$

Since however we have

$$(A_1, B_1) \equiv (A', B'),$$

it follows that: $(A_1, B_1) \equiv (A_2, B_2)$.

CASE V. *P_1 and P_2 lie in the same separation threefold W.*

If P_1 and P_2 have no element in common, then since they both lie in W they must be parallel to one another and the result follows from Case IV.

We shall therefore suppose that P_1 and P_2 have an element in common, but are distinct

Then, by Theorem 150, they have a second element in common, and therefore have a general line in common which we shall call b.

Let C be any element in b and let a_1 and a_2 be separation lines passing through C and normal to b and lying in P_1 and P_2 respectively.

Let B be an element in b such that:

$$(C_1, B_1) \equiv (C, B).$$

Then we shall also have

$$(C_2, B_2) \equiv (C, B).$$

Let A_1' and A_2' be elements in a_1 and a_2 respectively such that:

$$(C_1, A_1) \equiv (C, A_1')$$

and $(C_2, A_2) \equiv (C, A_2')$.

Then since $(C_1, A_1) \equiv (C_2, A_2)$,

we have $(C, A_1') \equiv (C, A_2')$.

Thus, by Case II, it follows that:

$$(A_1', B) \equiv (A_2', B) \quad \text{......(1).}$$

But, by Case IV, it follows that:

$$(A_1, B_1) \equiv (A_1', B) \quad \text{......(2),}$$

and similarly it follows that:

$$(A_2, B_2) \equiv (A_2', B) \quad \text{......(3).}$$

Thus from (1), (2) and (3) it follows that:

$$(A_1, B_1) \equiv (A_2, B_2).$$

Thus whether P_1 and P_2 are identical, or parallel, or whether they have a general line in common, the theorem holds provided P_1 and P_2 lie in the same separation threefold W.

CASE VI. *P_1 and P_2 do not lie in one separation threefold.*

In this case we may take one separation threefold, say W_1, which contains P_1 and another separation threefold, say W_2, which contains P_2.

Now W_2 may be either parallel to W_1, or else not parallel to it and, if not parallel, we know that W_1 and W_2 must have a general plane in common, which must obviously be a separation plane.

Suppose then first that W_2 is parallel to W_1 and let C be any element in W_2.

Then there is a general line, say a, passing through C and lying in W_2 which is parallel to C_1A_1.

Similarly there is a general line, say b, passing through C and lying in W_2 which is parallel to C_1B_1.

Thus, since B_1C_1 is normal to A_1C_1, it follows that b is normal to a.

Now let A and B be elements in a and b respectively such that:

$$(C_1, A_1) \mid\equiv\mid (C, A)$$

and $$(C_1, B_1) \mid\equiv\mid (C, B).$$

Then, by Theorem 180, we must have

$$(A_1, B_1) \mid\equiv\mid (A, B).$$

But now, since $(C_1, A_1) \equiv (C_2, A_2)$,

it follows that: $(C_2, A_2) \equiv (C, A)$,

and since $(C_1, B_1) \equiv (C_2, B_2)$,

it follows that: $(C_2, B_2) \equiv (C, B)$.

Thus since A, B and C lie in W_2 which also contains P_2, it follows by Case V that:

$$(A, B) \equiv (A_2, B_2).$$

Thus, since $(A_1, B_1) \equiv (A, B)$,

it follows that: $(A_1, B_1) \equiv (A_2, B_2)$.

Suppose next that W_2 is not parallel to W_1 and let S be the separation plane which they have in common.

Let a be any separation line in S, and C be any element in it.

Let b be the separation line which passes through C and lies in S and which is normal to a.

Let A and B be elements in a and b respectively such that:

$$(C_1, A_1) \equiv (C, A)$$

and $$(C_1, B_1) \equiv (C, B).$$

Then we shall also have

$$(C_2, A_2) \equiv (C, A)$$

and $$(C_2, B_2) \equiv (C, B).$$

But, since P_1 and S lie in W_1, it follows by Case V that:

$$(A_1, B_1) \equiv (A, B).$$

Also, since P_2 and S lie in W_2, it follows by Case V that:

$$(A_2, B_2) \equiv (A, B).$$

Thus we get finally: $(A_1, B_1) \equiv (A_2, B_2)$.

Combining now Cases V and VI we see that whether P_1 and P_2 lie in one separation threefold or not, the theorem still holds.

Thus the theorem holds in all cases.

Theorem 190

If A_1, B_1, C_1 be the corners of a triangle in a separation plane P_1 and A_2, B_2, C_2 be the corners of a triangle in a separation plane P_2, and if further

$$(C_1, A_1) \equiv (C_2, A_2),$$
$$(A_1, B_1) \equiv (A_2, B_2),$$

while $B_1 C_1$ is normal to $A_1 C_1$, and $B_2 C_2$ is normal to $A_2 C_2$, then we shall also have

$$(C_1, B_1) \equiv (C_2, B_2).$$

Let B_2' be an element in $B_2 C_2$ and on the same side of C_2 as is B_2 and such that:

$$(C_1, B_1) \equiv (C_2, B_2').$$

Then by Theorem 189 we must have

$$(A_1, B_1) \equiv (A_2, B_2'),$$

and so we must have $(A_2, B_2) \equiv (A_2, B_2')$.

Suppose now, if possible, that B_2' is distinct from B_2 and let O be the mean of B_2 and B_2'.

Then, by Theorem 187, $A_2 O$ must be normal to $B_2 C_2$.

But $A_2 C_2$ is normal to $B_2 C_2$ and so, since P_2 is a separation plane, we should have $A_2 O$ identical with $A_2 C_2$ and therefore O would be identical with C_2.

Since, however, O is supposed to be the mean of B_2 and B_2', it would require to be linearly between them and so B_2 and B_2' would be on opposite sides of C_2, contrary to hypothesis.

Thus the supposition that B_2 and B_2' are distinct leads to a contradiction and so B_2' must be identical with B_2.

But $(C_1, B_1) \equiv (C_2, B_2')$,

and therefore $(C_1, B_1) \equiv (C_2, B_2)$

as was to be proved.

THEOREM 191

If A_1, B_1, C_1 be the corners of a triangle in a separation plane P_1 and A_2, B_2, C_2 be the corners of a triangle in a separation plane P_2, and if further

$$(A_1, B_1) \equiv (A_2, B_2),$$
$$(A_1, C_1) \equiv (A_2, C_2),$$
$$(B_1, C_1) \equiv (B_2, C_2),$$

while A_1C_1 is normal to B_1C_1, then we shall also have A_2C_2 normal to B_2C_2.

Let a be a separation line passing through C_2 and lying in P_2 and which is normal to B_2C_2, and let A_2' be an element in a on the same side of B_2C_2 as is A_2 and such that:

$$(A_2', C_2) \equiv (A_1, C_1).$$

Then, by Theorem 189, we shall have

$$(A_2', B_2) \equiv (A_1, B_1).$$

It follows that we must have

$$(A_2, B_2) \equiv (A_2', B_2),$$

and further $\qquad (A_2, C_2) \equiv (A_2', C_2).$

Now if A_2' lies in A_2B_2 it must be identical with A_2 for there is only one element, say A, distinct from A_2 and lying in A_2B_2 and such that:

$$(A_2, B_2) \equiv (A, B_2),$$

and this element A lies on the opposite side of B_2 to that on which A_2 lies.

Thus, since A_2 and A_2' lie on the same side of B_2C_2 and therefore on the same side of B_2, it follows that A_2' must be identical with A_2.

Similarly if A_2' lies in A_2C_2 it must be identical with A_2.

Suppose now, if possible, that A_2' is distinct from A_2 and lies neither in A_2B_2 nor in A_2C_2, and let O be the mean of A_2 and A_2'.

Then, by Theorem 187, B_2O must be normal to A_2A_2' and similarly C_2O must be normal to A_2A_2'.

Thus, since B_2O and C_2O lie in the same separation plane as A_2A_2', it follows that B_2O and C_2O would be the same general line which accordingly would be identical with B_2C_2, and so O would require to lie in B_2C_2.

But, since O is supposed to be the mean of A_2 and A_2', it would have to be linearly between them and so, A_2A_2' being distinct from B_2C_2, we should have A_2 and A_2' on opposite sides of B_2C_2, contrary to hypothesis.

Thus the assumption that A_2' is distinct from A_2 leads to a contradiction and so A_2' is identical with A_2.

But $A_2'C_2$ is normal to B_2C_2 by hypothesis and so A_2C_2 is normal to B_2C_2 as was to be proved.

Theorem 192

If A_1, B_1, C_1 be the corners of a triangle in a separation plane P_1 and A_2, B_2, C_2 be the corners of a triangle in a separation plane P_2, and if further

$$(A_1, B_1) \equiv (A_2, B_2),$$
$$(A_1, C_1) \equiv (A_2, C_2),$$
$$(B_1, C_1) \equiv (B_2, C_2),$$

while N_1 is an element in B_1C_1 such that A_1N_1 is normal to B_1C_1, and N_2 is an element in B_2C_2 such that A_2N_2 is normal to B_2C_2; and if N_1 be distinct from both B_1 and C_1, then N_2 will be distinct from both B_2 and C_2, and we shall also have

$$(A_1, N_1) \equiv (A_2, N_2),$$
$$(B_1, N_1) \equiv (B_2, N_2),$$
$$(C_1, N_1) \equiv (C_2, N_2).$$

If N_1 be linearly between B_1 and C_1, let N_2' be an element in B_2C_2 and on the same side of B_2 as is C_2 and such that:

$$(B_1, N_1) \equiv (B_2, N_2').$$

Let C_2' be an element in B_2C_2 and on the opposite side of N_2' to that on which B_2 lies and such that:

$$(N_1, C_1) \equiv (N_2', C_2').$$

Then, by Theorem 182, we shall have

$$(B_1, C_1) \equiv (B_2, C_2'),$$

and so

$$(B_2, C_2) \equiv (B_2, C_2').$$

But C_2 and C_2' both lie on the same side of B_2 and so they must be identical.

Thus we must have

$$(C_1, N_1) \equiv (C_2, N_2').$$

Again, if C_1 be linearly between B_1 and N_1 let N_2' be an element in B_2C_2 and on the opposite side of C_2 to that on which B_2 lies and such that:

$$(C_1, N_1) \equiv (C_2, N_2').$$

Then, by Theorem 182, we shall have

$$(B_1, N_1) \equiv (B_2, N_2').$$

Similarly if B_1 be linearly between C_1 and N_1, let N_2' be an element in C_2B_2 and on the opposite side of B_2 to that on which C_2 lies and such that:

$$(B_1, N_1) \equiv (B_2, N_2').$$

Then, by Theorem 182, we shall have

$$(C_1, N_1) \equiv (C_2, N_2').$$

Thus in all three cases N_2' has been taken in B_2C_2 in such a manner that:

$$(B_1, N_1) \equiv (B_2, N_2')$$

and

$$(C_1, N_1) \equiv (C_2, N_2').$$

Now let a be a separation line lying in P_2 and passing through N_2' and normal to B_2C_2.

Let an element A_2' be selected in a and on the same side of B_2C_2 as A_2 lies and such that:

$$(N_1, A_1) \equiv (N_2', A_2').$$

Then, by Theorem 189, it follows that:

$$(A_1, B_1) \equiv (A_2', B_2)$$

and

$$(A_1, C_1) \equiv (A_2', C_2).$$

Thus we must have

$$(A_2, B_2) \equiv (A_2', B_2)$$

and

$$(A_2, C_2) \equiv (A_2', C_2).$$

Then, as in the last theorem, we may prove that the elements A_2 and A_2' must be identical and, since A_2N_2' is normal to B_2C_2 and intersects it in the element N_2' and, since P_2 is a separation plane, it follows that N_2' is identical with N_2, and therefore N_2 is distinct from both B_2 and C_2.

Thus we must have

$$(A_1, N_1) \equiv (A_2, N_2),$$
$$(B_1, N_1) \equiv (B_2, N_2),$$
$$(C_1, N_1) \equiv (C_2, N_2),$$

and so the theorem is proved.

It is also evident from the manner in which N_2' was determined that we must have

N_2 linearly between B_2 and C_2,
or C_2 linearly between B_2 and N_2,
or B_2 linearly between C_2 and N_2,
according as N_1 is linearly between B_1 and C_1,
or C_1 is linearly between B_1 and N_1,
or B_1 is linearly between C_1 and N_1.

Remarks

If B, C_1 and C_2 be three distinct elements in a separation plane which do not all lie in one separation line, and if

$$(B, C_1) \equiv (B, C_2)$$

while N_1 and N_2 are elements in BC_1 and BC_2 respectively, such that C_2N_1 is normal to BC_1 and C_1N_2 is normal to BC_2, then, if we make the restriction that BC_1 is not normal to BC_2, we may take the triangle whose corners are B, C_1 and C_2 and apply the results of the last theorem to the one triangle taken in two aspects.

Since BC_1 is not normal to BC_2, it follows that neither N_1 nor N_2 can be identical with B; so that we can speak of the pairs (B, N_1) and (B, N_2).

Further, we could not have both N_1 identical with C_1 and N_2 identical with C_2; for then we should have two intersecting separation lines: BC_1 and BC_2 both normal to one separation line C_1C_2 lying in the same separation plane with them and this, we know, is impossible.

Thus either N_1 is distinct from C_1, or N_2 is distinct from C_2 and, without essential loss of generality, we may suppose that N_1 is distinct from C_1.

If then, in the last theorem, we take

$$\begin{gathered} B_2 \text{ identical with } B_1, \\ A_1 \text{ identical with } C_2, \\ A_2 \text{ identical with } C_1, \end{gathered}$$

and write B for B_1 or B_2, we get N_2 distinct from C_2 and

$$\begin{gathered} (C_2, N_1) \equiv (C_1, N_2), \\ (B, N_1) \equiv (B, N_2), \\ (C_1, N_1) \equiv (C_2, N_2). \end{gathered}$$

Also the *linearly between* relations of B, N_2 and C_2 will be similar to those of B, N_1 and C_1 respectively.

Definitions. If O and X_0 be two distinct elements in a separation plane S, then the set of all elements in S such as X, where

$$(O, X) \equiv (O, X_0),$$

will be called a *separation circle*.

The element O will be called the *centre* of the separation circle.

Any one of the linear intervals such as OX will be called a *radius* of the separation circle.

If X_1 and X_2 be two elements of the separation circle such that

X_1X_2 passes through O, then the linear interval X_1X_2 will be called a *diameter* of the separation circle.

Any element which lies in a radius but which is not an element of the separation circle itself will be said to lie *inside* or in the *interior* of the separation circle.

Any element which lies in S but not in a radius will be said to lie *outside* or *exterior to* the separation circle.

THEOREM 193

(1) *If a separation line a and a separation circle both lie in the same separation plane S, and if any element A of a lies within the separation circle, then the latter has two elements in common with a and the element A lies linearly between them.*

(2) *If a separation line a has two elements D and E in common with a separation circle lying in a separation plane S, and if an element A of a lies linearly between D and E, then A lies within the separation circle.*

Consider the first part of the theorem.

Let O be the centre of the separation circle and let a separation line in S be taken through O and A. Then there are two elements of this separation line, say X_1 and X_2, which lie on the circle and are therefore such that:

$$(O, X_1) \equiv (O, X_2),$$

and A must lie linearly between X_1 and X_2.

Let c be an inertia line through O normal to S and let C be the element common to c and the α sub-set of X_1.

Then CX_1 is an optical line and, since X_2 is also an element of the circle, we must also have CX_2 an optical line.

Further, C is *after* X_1 and since X_1X_2 is a separation line, we must also have C *after* X_2.

Now, by Theorem 73, since A is linearly between X_1 and X_2, it follows that CA is an inertia line and A is *before* C.

But, since CA is an inertia line which intersects the separation line a, it follows that CA and a lie in an inertia plane, say P.

Then there are two optical lines passing through C and lying in P and these will intersect a in two distinct elements, say D and E.

Then, since c is normal to S, it follows that:

$$(O, D) \equiv (O, E) \equiv (O, X_1) \equiv (O, X_2),$$

and so D and E lie on the separation circle and accordingly, since they also lie in a, the existence of the two elements is proved.

It also follows that A must lie linearly between D and E. For, in the first place, A could not coincide with either D or E, since CD and CE are optical lines while CA is an inertia line. Again, D could not lie linearly between A and E for, since C is *after* both A and E, it would follow, by Theorem 73, that CD must be an inertia line; which is impossible. Similarly E could not lie linearly between A and D. It remains that A must lie linearly between D and E; which proves the first part of the theorem.

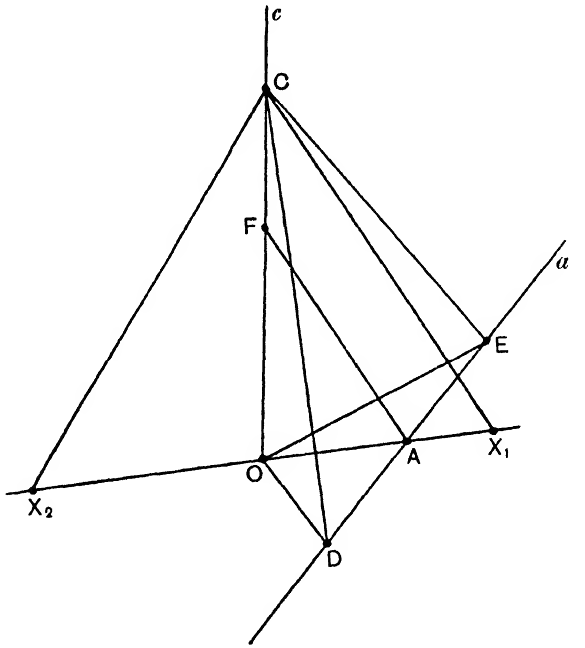


Fig. 48.

Consider now the second part of the theorem.

As before, let O be the centre of the separation circle and let c be an inertia line through O normal to S.

Let C be the element common to c and the α sub-set of D. Then CD is an optical line and C is *after* D.

Also, since D and E are both elements of the circle centre O, it follows that CE is also an optical line and, since DE is a separation line, we must also have C *after* E.

But now, since by hypothesis A is linearly between D and E, it follows, by Theorem 73, that CA is an inertia line and A is *before* C.

Now, in the very special case where A is identical with O, it is

obvious that A lies within the separation circle; so we shall suppose that A is not identical with O.

Let F be the element common to c and the α sub-set of A. Then FA is an optical line and F is *after* A.

But since OA is a separation line and FO is an inertia line we must also have F *after* O.

Now F could not be identical with C since FA is an optical line while CA is an inertia line.

Also F could not be *after* C for then we should have C *after* A and *before* F: two distinct elements of the optical line AF in which C does not lie, and we know that this is impossible.

It remains that C is *after* F so that F is linearly between O and C.

Now, since c is an inertia line, therefore c and OA lie in an inertia plane, and, if we take an optical line through C parallel to FA, it will intersect the separation line OA in some element X_1.

Then X_1 will be an element of the circle and, since F is linearly between O and C, it follows that A is linearly between O and X_1 and so A lies within the separation circle.

Thus the second part of the theorem is proved.

Theorem 194

If A, B and C be the corners of a triangle in a separation plane, and if BC be normal to AC, then the side AB is greater than either of the other two sides of the triangle.

It will be sufficient to prove that AB is greater than AC.

Let D be an element in BC such that C is the mean of B and D.

Then, by Theorem 186, we must have

$$(A, D) \equiv (A, B);$$

and so B and D are two elements of a separation circle of centre A; while the separation line BD has the two elements B and D in common with it.

Further, the element C lies linearly between B and D and so, by Theorem 193, C lies within the circle.

Thus AB is greater than AC; and similarly we may prove that AB is greater than BC.

Remarks

If a separation line a and a separation circle both lie in a separation plane S, they can either have no element in common, or one element in common, or two elements in common, but cannot have more than two.

Taking O as centre, let a separation line passing through O, lying in S and normal to a intersect a in an element N and let B be any element of a distinct from N.

Then, by Theorem 194, OB is greater than ON, so that if ON is greater than a radius of the circle, then OB is always greater than a radius and a can have no element in common with the circle.

If ON is equal to a radius, then the element N lies on the circle but B cannot do so and, in this case, a has one element in common with the circle.

If ON is less than a radius, then N must lie within the circle, so that, by the first part of Theorem 193, a must have two elements in common with it, say D and E.

If O, D and E should happen to lie in one separation line, N would coincide with O and would therefore be the mean of D and E; while if O, D and E do not lie in one separation line the same result follows by Theorem 188.

Now suppose, if possible, that there is an element E' distinct from D and E and common to the circle and the separation line a. Then N would require to be the mean of D and E' as well as of D and E, which is impossible by Theorem 62.

Thus no such element as E' can exist and so the separation circle cannot have more than two elements in common with the separation line a.

It follows very simply from Theorem 194 that: *If a triangle lies in a separation plane, then the sum of the lengths of any two sides is greater than that of the third side.*

This may be shown as follows:

Let A, B, C be the corners of any triangle in a separation plane S and let a separation line through A normal to BC intersect BC in an element N.

Then (1) if N is linearly between B and C we have BA is greater than BN and AC is greater than NC so that

$$BA + AC \text{ is greater than } BC.$$

(2) If N coincides with C we have BA is greater than BC and so

$$BA + AC \text{ is greater than } BC.$$

Similarly, if N coincides with B, we have AC greater than BC and so

$$BA + AC \text{ is greater than } BC.$$

(3) If C is linearly between B and N we have BA is greater than BN while BN is greater than BC and so BA is greater than BC; from which it follows that:

$$BA + AC \text{ is greater than } BC.$$

Similarly, if B is linearly between C and N, we may prove the same result.

These cover all the possibilities which are open; so that in all cases we have

$$BA + AC \text{ is greater than } BC.$$

By a similar method we may prove that the sum of the lengths of any other two sides of the triangle is greater than that of the third side.

Another important result which can readily be obtained, using the notation of Theorem 194, is as follows:

If the separation line through C normal to AB intersects AB in M, then M is linearly between A and B.

For we have AB is greater than BC and BC is greater than BM, so that AB is greater than BM. Similarly AB is greater than AM. Thus, since M lies in AB, it must be linearly between A and B.

Definition. If A, B and C be the corners of a triangle in a separation or inertia plane and if BC be normal to AC, then the side AB will be called the *hypotenuse* of the triangle.

In case the triangle lies in an optical plane and BC be normal to AC, then either BC or AC must be an optical line and, whichever it be, that one must also be normal to AB.

Thus, when the triangle lies in an optical plane, two of its sides would equally well be entitled to the name *hypotenuse*. This could never be the case either in a separation or in an inertia plane.

Angle boundaries in separation planes

We are now going to make three successive applications of the result proved in the remarks at the end of Theorem 192 to the case of a construction obtained from two intersecting separation lines lying in a separation plane and which are not normal to one another.

Let the separation lines be called $\bar{x}_1$ and $\bar{x}_2$ and let O be their element of intersection.

Then O will divide the separation line $\bar{x}_1$ into two half-lines, which we shall denote by x_1 and x_1'; while it will divide the separation line $\bar{x}_2$ into two half-lines which we shall denote by x_2 and x_2'.

Let C_1, C_1', C_2, C_2' be any elements in x_1, x_1', x_2, x_2' respectively such that

$$(O, C_1) \equiv (O, C_1') \equiv (O, C_2) \equiv (O, C_2'),$$

and let separation lines through C_2 and C_2' normal to the separation line $\bar{x}_1$ intersect it in N_{21} and N'_{21} respectively; while separation lines through C_1 and C_1' normal to the separation line $\bar{x}_2$ intersect it in N_{12} and N'_{12} respectively.

Then, since by hypothesis, $\bar{x}_1$ and $\bar{x}_2$ are supposed not to be normal to one another, it follows that none of the elements N_{21}, N'_{21}, N_{12}, N'_{12} can coincide with O.

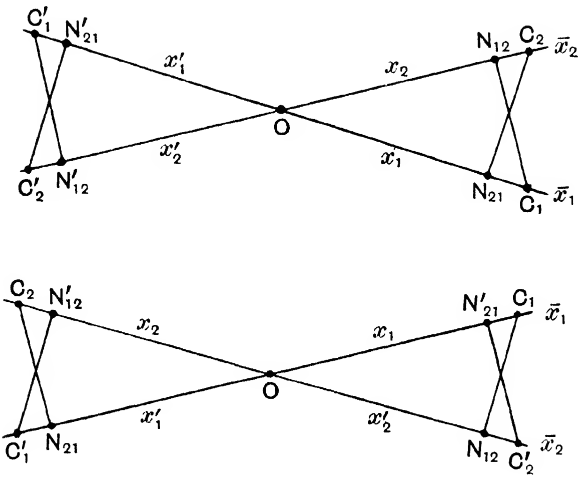


Fig. 49.

If we consider first the element N_{21}, then since, by Theorem 194, the segment ON_{21} is less than the segment OC_2, it follows that N_{21} must either lie linearly between O and C_1, or else linearly between O and C_1'.

Without any essential loss of generality we may suppose that N_{21} is linearly between O and C_1, so that N_{21} lies in x_1.

Then it follows that N_{12} must be linearly between O and C_2, so that N_{12} lies in x_2 and we must also have

$$(C_2, N_{21}) \equiv (C_1, N_{12})$$

and
$$(O, N_{21}) \equiv (O, N_{12}).$$

Now, since N_{12} lies in x_2 while C_2' lies in x_2', it follows that O is linearly between C_2' and N_{12}.

Thus, by a second application of the theorem, it follows that O is linearly between C_1 and N'_{21}, so that N'_{21} lies in x_1', while

$$(C_1, N_{12}) \equiv (C_2', N'_{21})$$
$$(O, N_{12}) \equiv (O, N'_{21}).$$

Now, since N'_{21} lies in x_1' and since the segment ON'_{21} must be less than the segment OC_2', it follows that N'_{21} must be linearly between O and C_1'.

Thus, by a third application of the theorem, it follows that N'_{12} must be linearly between O and C_2', so that N'_{12} must lie in x_2', while

$$(C_2', N'_{21}) \equiv (C_1', N'_{12})$$

and

$$(O, N'_{21}) \equiv (O, N'_{12}).$$

Thus we must have

$$(C_2, N_{21}) \equiv (C_1, N_{12}) \equiv (C_2', N'_{21}) \equiv (C_1', N'_{12}),$$

and

$$(O, N_{21}) \equiv (O, N_{12}) \equiv (O, N'_{21}) \equiv (O, N'_{12});$$

while

$$N_{21} \text{ lies in } x_1,$$
$$N_{12} \text{ lies in } x_2,$$
$$N'_{21} \text{ lies in } x_1',$$
$$N'_{12} \text{ lies in } x_2'.$$

The congruences hold whether N_{21} lies in x_1 or x_1', but, if N_{21} lies in x_1', we shall have instead of the above

$$N_{21} \text{ lies in } x_1',$$
$$N'_{12} \text{ lies in } x_2,$$
$$N'_{21} \text{ lies in } x_1,$$
$$N_{12} \text{ lies in } x_2'.$$

Now any two separation lines in the separation plane which are normal to $\bar{x}_1$ must be parallel to one another, and similarly any two which are normal to $\bar{x}_2$ must be parallel to one another.

Accordingly if we take different positions for C_1, C_2, C_1', C_2' in the corresponding half-lines x_1, x_2, x_1', x_2' we may apply the results given in the remarks at the end of Theorem 185 and show that the ratios of the segments,

$$ON_{21} : OC_2,$$
$$C_2N_{21} : OC_2,$$
$$C_2N_{21} : ON_{21},$$

are independent of the position of C_2 in the half-line x_2, and we get corresponding constancy of ratios for all positions of C_1 in x_1, of C_1' in x_1' and of C_2' in x_2'; so that these ratios are definite for any definite pair of such separation lines.

Again let $\bar{O}$, $\bar{C}$ and $\bar{N}$ be the corners of a triangle lying in any separation plane and let $\bar{C}\bar{N}$ be normal to $\bar{O}\bar{N}$.

(i) Suppose, in the first place that

$$\bar{O}\bar{C}=OC_2$$

and that $$\bar{O}\bar{N}:\bar{O}\bar{C}=ON_{21}:OC_2.$$

Then $$\bar{O}\bar{N}:ON_{21}=\bar{O}\bar{C}:OC_2.$$

Thus, since $\bar{O}\bar{C}=OC_2$, it follows from the criterion of proportion that $\bar{O}\bar{N}=ON_{21}$ and, by Theorem 190, we must have

$$\bar{C}\bar{N}=C_2N_{21}.$$

It follows that

$$\bar{C}\bar{N}:\bar{O}\bar{C}=C_2N_{21}:OC_2$$

and $$\bar{C}\bar{N}:\bar{O}\bar{N}=C_2N_{21}:ON_{21}.$$

(ii) Next let us suppose that

$$\bar{O}\bar{C}=OC_2$$

and that $$\bar{C}\bar{N}:\bar{O}\bar{C}=C_2N_{21}:OC_2.$$

Then again, since $\bar{O}\bar{C}=OC_2$, it follows that $\bar{C}\bar{N}=C_2N_{21}$ and accordingly, by Theorem 190, it follows that

$$\bar{O}\bar{N}=ON_{21}.$$

Thus we see that

$$\bar{O}\bar{N}:\bar{O}\bar{C}=ON_{21}:OC_2$$

and $$\bar{C}\bar{N}:\bar{O}\bar{N}=C_2N_{21}:ON_{21}.$$

(iii) Suppose finally that

$$\bar{O}\bar{N}=ON_{21}$$

and that $$\bar{C}\bar{N}:\bar{O}\bar{N}=C_2N_{21}:ON_{21}.$$

Then we shall have $\bar{C}\bar{N}=C_2N_{21}$ and since also $\bar{O}\bar{N}=ON_{21}$ it follows, by Theorem 189, that

$$\bar{O}\bar{C}=OC_2.$$

Thus we see that

$$\bar{O}\bar{N}:\bar{O}\bar{C}=ON_{21}:OC_2$$

and $$\bar{C}\bar{N}:\bar{O}\bar{C}=C_2N_{21}:OC_2.$$

Thus we see that if any one of the three proportionalities:

$$\bar{O}\bar{N}:\bar{O}\bar{C}=ON_{21}:OC_2,$$
$$\bar{C}\bar{N}:\bar{O}\bar{C}=C_2N_{21}:OC_2,$$
$$\bar{C}\bar{N}:\bar{O}\bar{N}=C_2N_{21}:ON_{21},$$

holds, then the remaining two will also hold, and the pair of separation lines $\bar{O}\bar{N}$ and $\bar{O}\bar{C}$ will be characterised by the same triplet of ratios as were the original pair of separation lines: $\bar{x}_1$ and $\bar{x}_2$.

We shall find it convenient to denote these ratios by special names as follows:

The ratio $$ON_{21} : OC_2$$
we shall call the c-ratio of the separation lines $\bar{x}_1$ and $\bar{x}_2$; the ratio
$$C_2N_{21} : OC_2$$
we shall call the s-ratio; while the ratio
$$C_2N_{21} : ON_{21}$$
we shall call the t-ratio of the separation lines. The letters c, s and t are the initial letters of the words cosine, sine and tangent respectively, but, since they are ratios of absolute magnitudes, and the word "ratio" is used in the Euclidean sense, there is no question of sign involved.

Employing the above notation, but removing the restriction that the separation lines $\bar{x}_1$ and $\bar{x}_2$ are not normal and also any implied restriction that they are necessarily distinct, we may introduce the following definitions:

Definition. If x_1 and x_2 be separation half-lines lying in a separation plane S and having a common end O, then, together with the element O, they will be said to form an *angle-boundary* and the element O will be called its *vertex* while x_1 and x_2 will be called its *sides*.

For the sake of brevity we shall frequently speak of the sides as forming the angle-boundary, without explicit mention of the vertex.

If a separation line in S taken through any element of x_2 normal to the separation line $\bar{x}_1$ intersects the latter in an element of x_1, then the angle-boundary which x_1 and x_2 form will be said to be *acute*.

If such a separation line intersects the separation line $\bar{x}_1$ in an element of x_1', then the angle-boundary which x_1 and x_2 form will be said to be *obtuse*.

If such a separation line intersects the separation line $\bar{x}_1$ in the element O, then the angle-boundary which x_1 and x_2 form will be said to be *right*.

This is equivalent to saying that x_1 and x_2 will form a right angle-boundary provided that the separation lines $\bar{x}_1$ and $\bar{x}_2$ are normal to one another.

If x_1 and x_2 form two distinct portions of the same separation line they will be said to form a *straight* or *flat* angle-boundary; while if x_1 and x_2 are identical they will be said to form a *null* angle-boundary.

It will be observed from the results obtained above that if x_1 and x_2 form an acute angle-boundary, then x_1' and x_2' will also form an acute

angle-boundary, while x_1' and x_2 will form an obtuse angle-boundary; as will also x_1 and x_2'.

It will be observed that we may interchange the rôles of x_1 and x_2 in our definitions without affecting the acute, right or obtuse character of the angle-boundary formed by them.

In case x_1 and x_2 form a right angle-boundary, then so also will x_1' and x_2', x_1 and x_2', x_2 and x_1'.

Definition. The angle-boundary formed by x_2 and x_1' will be called a *supplement* of the angle-boundary formed by x_2 and x_1 and conversely.

We shall speak of the c-ratio, s-ratio or t-ratio of an angle-boundary meaning thereby the c-ratio, s-ratio or t-ratio of the complete pair of separation lines of which the sides of the angle-boundary are parts.

Definition. Angle-boundaries which are not right angle-boundaries will be said to be *congruent* provided that their c-ratios are equal and their acute or obtuse characters are the same.

Definition. Any right angle-boundary will be said to be *congruent* to any right angle-boundary.

It is to be noted that the *congruency of angle-boundaries*, as here defined, is a similarity in the relationships of the pairs of half-lines which form the sides of the angle-boundaries which are said to be congruent.

It does not, in itself, imply more than this; and certain other things have to be taken into consideration before one can adequately treat such theorems as involve the "*addition of angles*" in separation planes.

The customary notation for an "angle", such as $\angle ABC$, is only properly applicable to the relationship which the pair of half-lines BA and BC stand in to one another, and, although the notation is continually employed in ordinary geometry to represent an angular magnitude, it cannot, strictly speaking, do so without ambiguity.

We shall accordingly make use of the notation $\angle ABC$ to denote what we have called an "angle-boundary" whose sides are the half-lines BA and BC, and shall denote the congruence of angle-boundaries by the symbol $\equiv$ placed between the symbols for the latter.

According to the above definitions a null angle-boundary is to be regarded as acute, while a flat angle-boundary is to be regarded as obtuse.

We have employed the c-ratios in the definition of the congruence of acute or obtuse angle-boundaries, but we might also have used either

the s-ratios or the t-ratios; were it not that we shall find the c-ratios more convenient when we come to introduce numerical measurement with + and − signs. It will then appear that the acute or obtuse character of the angle-boundary may be expressed by the sign of the cosine, but not by that of the sine or tangent.

It will be observed that, as a result of our definitions along with the congruence relations already proved, it follows that the angle-boundary made by x_1 and x_2 is congruent to the angle-boundary made by x_1' and x_2', while the angle-boundary made by x_1 and x_2' is congruent to the angle-boundary made by x_1' and x_2.

We are now in a position to prove various theorems involving angle-boundaries in separation planes.

It will be observed that the results of Theorem 192 enable us at once to write

$$B_1N_1 : B_1A_1 = B_2N_2 : B_2A_2$$

and

$$C_1N_1 : C_1A_1 = C_2N_2 : C_2A_2,$$

and the ordinal relations of B_1, N_1 and C_1 are in all cases similar to those of B_2, N_2 and C_2 respectively: so that if N_1 be distinct from both B_1 and C_1 we have

$$\angle A_1B_1C_1 \equiv \angle A_2B_2C_2,$$

and

$$\angle A_1C_1B_1 \equiv \angle A_2C_2B_2.$$

If N_1 coincides with B_1 then we know that N_2 must coincide with B_2, which merely means that $\angle A_1B_1C_1$ and $\angle A_2B_2C_2$ are both right and therefore are congruent and we still have

$$C_1N_1 : C_1A_1 = C_2N_2 : C_2A_2,$$

so that

$$\angle A_1C_1B_1 \equiv \angle A_2C_2B_2.$$

If N_1 coincides with C_1 we obtain analogous results.

Thus in all cases we have

$$\angle A_1B_1C_1 \equiv \angle A_2B_2C_2,$$
$$\angle A_1C_1B_1 \equiv \angle A_2C_2B_2,$$

and by a similar method we can show that

$$\angle B_1A_1C_1 \equiv \angle B_2A_2C_2.$$

Thus we see that *if A_1, B_1, C_1 be the corners of a triangle in a separation plane P_1 and A_2, B_2, C_2 be the corners of a triangle in a separation plane P_2, and if further*

$$A_1B_1 = A_2B_2, \quad B_1C_1 = B_2C_2, \quad C_1A_1 = C_2A_2,$$

then corresponding angle-boundaries in the two triangles will be congruent.

Another very important result which follows very simply is this:

If two parallel separation lines a and b lying in a separation plane be intersected by another separation line c in the elements A and B respectively, and if further F, be any element of c such that A is linearly between B and F, and if A_1 and B_1 be any elements of a and b respectively which both lie on the same side of c, we shall have

$$\angle A_1AF \equiv \angle B_1BF.$$

In the special case where c is normal to a, it is also normal to b and the angle-boundaries are both right and therefore are congruent.

If c be not normal to a, let a separation line through F in the separation plane be taken normal to a and let it intersect a in A' and b in B'.

Then, as we saw in the remarks at the end of Theorem 185,

$$AA' : FA = BB' : FB,$$

and so, since $\angle$'s $A'AF$ and $B'BF$ are both acute, we have

$$\angle A'AF \equiv \angle B'BF.$$

Also, since A is linearly between B and F, we have A' linearly between B' and F, so that A' and B' are both on the same side of c.

If this should happen to be the same side of c as A_1 and B_1 lie on, then $\angle A_1AF$ is identical with $\angle A'AF$ while $\angle B_1BF$ is identical with $\angle B'BF$, so that

$$\angle A_1AF \equiv \angle B_1BF.$$

If, on the other hand, A' and B' should happen to be on the opposite side of c to that on which A_1 and B_1 lie, then we should have $\angle A_1AF$ the supplement of $\angle A'AF$ and $\angle B_1BF$ the supplement of $\angle B'BF$, so that again we have

$$\angle A_1AF \equiv \angle B_1BF.$$

Thus the result holds in all cases.

Again, no matter whether $\angle A_1AF$ be right, acute or obtuse, let A_2 be any element of a such that A is linearly between A_1 and A_2.

Then we already know that:

$$\angle A_1AF \equiv \angle BAA_2,$$

and so we have $$\angle B_1BA \equiv \angle BAA_2:$$

another important result.

THEOREM 195

If A_1, B_1, C_1 be the corners of a triangle in a separation plane P_1, while A_2, B_2, C_2 are the corners of a triangle in a separation plane P_2 and if further

$$B_1A_1 = B_2A_2,$$
$$B_1C_1 = B_2C_2,$$
$$\angle A_1B_1C_1 \equiv \angle A_2B_2C_2,$$

then we shall also have

$$A_1 C_1 = A_2 C_2,$$
$$\angle B_1 C_1 A_1 \equiv \angle B_2 C_2 A_2,$$
$$\angle C_1 A_1 B_1 \equiv \angle C_2 A_2 B_2.$$

In case $\angle A_1 B_1 C_1$ and $\angle A_2 B_2 C_2$ should happen to be right we have already, by Theorem 189, $A_1 C_1 = A_2 C_2$, and, since the other angle-boundaries are all acute, and since we have

$$A_1 B_1 : A_1 C_1 = A_2 B_2 : A_2 C_2,$$

and
$$C_1 B_1 : C_1 A_1 = C_2 B_2 : C_2 A_2,$$

it follows that:
$$\angle B_1 C_1 A_1 \equiv \angle B_2 C_2 A_2,$$

and
$$\angle C_1 A_1 B_1 \equiv \angle C_2 A_2 B_2.$$

Next take the cases where $\angle A_1 B_1 C_1$ and $\angle A_2 B_2 C_2$ are both acute or both obtuse and let a separation line through C_1, lying in P_1 and normal to $A_1 B_1$ intersect $A_1 B_1$ in N_1; while a separation line through C_2, lying in P_2 and normal to $A_2 B_2$ intersects $A_2 B_2$ in N_2.

Then, since $\angle A_1 B_1 C_1 \equiv \angle A_2 B_2 C_2$, we have

$$B_1 N_1 : B_1 C_1 = B_2 N_2 : B_2 C_2,$$

and
$$C_1 N_1 : B_1 C_1 = C_2 N_2 : B_2 C_2.$$

Thus, since $B_1 C_1 = B_2 C_2$, we have

$$B_1 N_1 = B_2 N_2,$$

and
$$C_1 N_1 = C_2 N_2.$$

If $\angle A_1 B_1 C_1$ and $\angle A_2 B_2 C_2$ are both acute A_1 and N_1 will lie on the same side of B_1, while A_2 and N_2 will lie on the same side of B_2.

If $B_1 N_1$ should happen to equal $B_1 A_1$ then N_1 would coincide with A_1 while N_2 would coincide with A_2 and so, in this case, we should have

$$A_1 C_1 = A_2 C_2.$$

If $B_1 N_1$ should happen to be less than $B_1 A_1$ we should have N_1 linearly between B_1 and A_1 and also N_2 linearly between B_2 and A_2 and we should also have

$$N_1 A_1 = N_2 A_2.$$

If $B_1 N_1$ should happen to be greater than $B_1 A_1$ we should have A_1 linearly between B_1 and N_1 and also A_2 linearly between B_2 and N_2 and again we should have

$$N_1 A_1 = N_2 A_2.$$

Finally, if $\angle A_1 B_1 C_1$ and $\angle A_2 B_2 C_2$ are both obtuse we should have B_1 linearly between A_1 and N_1 and also B_2 linearly between A_2 and N_2 and once more we should have

$$N_1 A_1 = N_2 A_2.$$

Thus in all three cases, by Theorem 189, we have

$$A_1C_1 = A_2C_2.$$

It then follows by the result proved on p. 340 that:

$$\angle B_1C_1A_1 \equiv \angle B_2C_2A_2,$$

and

$$\angle C_1A_1B_1 \equiv \angle C_2A_2B_2,$$

and so the theorem is proved.

Theorem 196

If A and B be the extremities of a diameter of a separation circle lying in a separation plane and if C be any other element in the circumference of the separation circle, then AC is normal to BC.

Let O be the centre of the separation circle. Then O is the mean of B and A.

Let D be the mean of B and C.

Then we have

$$(O, B) \equiv (O, C)$$

and so, by Theorem 187, OD is normal to CB.

But, since O is the mean of B and A, while D is the mean of B and C, it follows that OD is parallel to AC.

Thus, since OD is normal to CB, we must also have AC normal to CB, as was to be proved.

Definition. If O and X_0 be two distinct elements in a separation threefold W, then the set of all elements in W such as X where

$$(O, X) \equiv (O, X_0)$$

will be called a *separation sphere.*

The element O will be called the *centre* of the separation sphere.

The terms *radius, diameter, inside, outside,* etc. may be defined in a similar manner to the case of a separation circle.

Remarks

As we have already pointed out, any element B in a general line a divides the remaining elements of a into two sets.

In case a is an optical line or an inertia line the two sets consist of those elements of a which are *before* B and those which are *after* B; but, in case a is a separation line, the sets are not capable of definition in quite so simple a form.

Confining our attention in the meantime to the cases where a is an optical line or an inertia line, we observe that we may group the element B itself with either of these sets.

Thus, if we divide all the elements of a into all those elements which are *before* B and all those elements which are not *before* B; then B itself is grouped with those which are not *before* B. If, on the other hand, we divide all the elements of a into all those elements which are *after* B and all those elements which are not *after* B; then B itself is grouped with those which are not *after* B.

In either case all the elements of a are divided into two sets such that every element of the one (which we may call the lower) set is *before* every element of the other (which we may call the upper) set.

The division is made by means of an element B which is explicitly mentioned.

The question arises as to whether it is possible to have a division of all the elements of a into two sets such that every element of the one set is *before* every element of the other set without the existence of an element making the division in the manner that B does in the above.

Although it seems reasonable to suppose that there should always be an element of this character; yet it appears that there is nothing in the postulates hitherto given which ensures that this must be the case.

These postulates do imply the existence of segments which bear certain incommensurable ratios to one another (as for example, the side and diagonal of a square), but these are only a restricted class of such ratios, and there are other incommensurable ratios conceivable whose existence is not thus implied.

In order to admit of such possibilities, we shall now give the final postulate of our system which is equivalent to the Axiom of Dedekind.

Postulate XXI. **If all the elements of an optical line be divided into two sets such that every element of the first set is before every element of the second set, then there is one single element of the optical line which is not before any element of the first set and is not after any element of the second set.**

Since an element is neither *before* nor *after* itself, it is evident that this one single element may belong either to the first or second set.

Again, if a be an optical line in an inertia plane P, then through each element of a there passes one single generator of P of the opposite system to that to which a belongs.

Also every such generator intersects a.

Thus there is a one-to-one correspondence between the elements of a and the generators of P of the other system and so it follows that: *if either system of generators of an inertia plane be divided into two sets such*

that every generator of the first set is a before-parallel of every generator of the second set, then there is one single generator of the system which is not a before-parallel of any generator of the first set and is not an after-parallel of any generator of the second set.

Again if b be any inertia or separation line and if P be an inertia plane containing it, then if we select either system of generators of P, there is a one-to-one correspondence between the elements of b and the generators of the selected system which pass through these elements.

If b be an inertia line and X and Y be any two elements of b, then X will be *before* or *after* Y according as the generator through X is a before- or after-parallel of that through Y.

Thus the property formulated in Post. XXI holds for an inertia line as well as for an optical line.

It is also clear that a corresponding result holds in the case of a separation line, but since here no element is either *before* or *after* another, the property must be formulated somewhat differently.

In order to state the result when b is a separation line we may make a perfectly arbitrary convention with regard to the use of the words *right* and *left*.

Thus if X and Y be any two elements of b, we may say that X is *to the left* or *right* of Y according as the generator of the selected system which passes through X is a before- or after-parallel of that through Y.

We may therefore state the property as follows:

If all the elements of a separation line be divided into two sets such that every element of the first set is to the left of every element of the second set, then there is one single element of the separation line which is not to the left of any element of the first set and is not to the right of any element of the second set.

With the introduction of the equivalent of the Dedekind axiom we have now reached the stage where we are in a position to set up a one-to-one correspondence between the elements of a general line l and the aggregate of *real numbers*.

Thus, if A_0 and A_1 be two distinct elements in l, it may be shown that there are elements $A_2, A_3, A_4, \dots A_n \dots$ in l and on the same side of A_0 as is A_1 and such that the segment A_0A_n is equal to n times the segment A_0A_1.

Similarly there are elements $A_{-1}, A_{-2}, A_{-3}, \dots A_n \dots$ lying in l but on the opposite side of A_0 and such that the segment $A_{-n}A_0$ is equal to n times the segment A_0A_1.

Again, it may easily be shown that corresponding to any positive rational number $r=\frac{p}{q}$ there is an element A_r in l and on the same side of A_0 as is A_1 and such that q times the segment A_0A_r is equal to p times the segment A_0A_1.

Similarly, corresponding to any negative rational number $-r=-\frac{p}{q}$; it may be shown that there is an element A_{-r} in l, but on the opposite side of A_0 and such that q times the segment $A_{-r}A_0$ is equal to p times the segment A_0A_1.

By making use of our equivalents of the axioms of Archimedes and Dedekind for the elements of l along with the corresponding properties of *real numbers*, it is possible to set up the one-to-one correspondence mentioned above.

The logical steps involved in setting up such a correspondence have been carefully investigated by others and it is unnecessary to go into further details here.

These may be found, for instance, in Pierpont's *Theory of Functions of Real Variables*, vol. I, chapters I and II, and in other works.

The absolute value of the difference of the real numbers corresponding to the two ends of any segment of l gives us a real number which may be called *the numerical value of the length* of the segment in terms of the unit segment A_0A_1.

If l be an inertia or separation line, the length of any segment of a general line of the same kind as l is always expressible in terms of our selected segment; but if l be an optical line we must restrict the meaning of the words "*of the same kind*" to co-directional optical lines.

THEOREM 197

If A, B, C be the corners of a triangle in a separation plane such that CA is normal to BA, and if a separation line through A normal to BC intersects BC in M, then

$$\angle BAM \equiv \angle BCA,$$

and

$$\angle CAM \equiv \angle CBA.$$

Since CA is normal to BA, therefore, by Theorem 194, CB is greater than CA and so if we take an element D in the half-line CA such that $(C, D) \equiv (C, B)$ we shall have A linearly between C and D.

If through D we take a separation line parallel to AM and meeting CB in N, then DN must also be normal to CB.

Then, as already seen, we shall have

$$(D, N) \equiv (B, A).$$

But, since DN is parallel to AM and D lies in the half-line CA, therefore, by Theorem 185,

$$AM : DN = AC : DC,$$

or

$$AM : AB = AC : BC.$$

Thus, since $\angle BAM$ and $\angle BCA$ are both acute, we have

$$\angle BAM \equiv \angle BCA.$$

Similarly

$$\angle CAM \equiv \angle CBA.$$

THEOREM 198

If A, B, C be the corners of a triangle in a separation plane and such that CA is normal to BA, then the square of the length of the side CB is equal to the sum of the squares of the lengths of the other two sides.

With C as centre and CB as radius take a circle in the separation plane and let it intersect the separation line CA in D and E.

Now, by Theorem 194, CA must be less than the radius of the circle and so A must be either linearly between C and E, or else linearly between C and D.

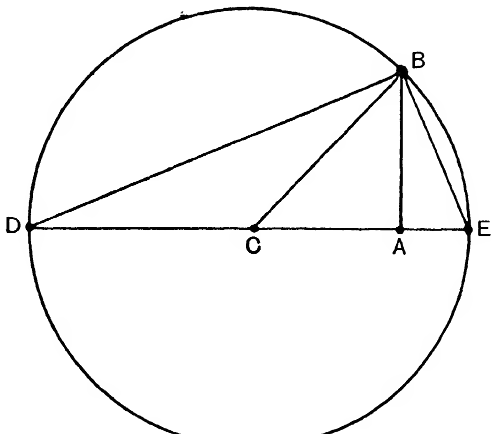


Fig. 50.

Without essential loss of generality we may suppose that A is linearly between C and E.

Now, from Theorem 196, we know that DB is normal to EB and so, by Theorem 197,

$$\angle EBA \equiv \angle EDB.$$

Taking the t-ratios of these, we have

$$AE : BA = BA : DA.$$

But $AE = CB - CA$

and $DA = CB + CA$

and therefore $(CB - CA) : BA = BA : (CB + CA)$.

If then we take any unit of length of a separation line, and let BA, CA and CB now represent the numerical values of these lengths in terms of the selected unit we get

$$CB^2 - CA^2 = BA^2,$$

or

$$CB^2 = BA^2 + CA^2,$$

as was to be proved.

This result is the equivalent of the "Theorem of Pythagoras": which accordingly holds in a separation plane.

Let us now, as we have previously done, denote the complete separation line of which x_1 and x_2 are parts by the symbols $\bar{x}_1$ and $\bar{x}_2$ respectively, and let the element O be the common end of x_1 and x_2.

Let any convenient unit of length be selected and let the element O be associated with the real number 0 and let elements in x_1 be associated with the positive real numbers representing their distances from O.

Let elements in x_1' be also associated with real numbers representing their distances from O, but having negative signs. We shall suppose the elements in the separation line $\bar{x}_2$ to be treated in an analogous manner.

If P be the element in x_2 which is at unit distance from O and if a normal through P on the separation line $\bar{x}_1$, intersects the latter in the element N, then the real number associated with N in $\bar{x}_1$ is characteristic of all angle-boundaries which are congruent to that made by x_2 and x_1.

Thus, when N is distinct from O, the absolute value of this real number represents the c-ratio $ON : OP$ and its sign is positive for acute angle-boundaries and negative for obtuse angle-boundaries; while, if N coincides with O, the real number associated with N is 0, which is characteristic of the case where we are dealing with right angle-boundaries.

It will be evident that the real number obtained in this way will be the cosine of an angle which x_2 makes with x_1, but, from the strict logical standpoint, we are not quite in a position to make this identification; since we have not as yet considered angles as distinguished from angle-boundaries. As, however, we require a notation for this function, we shall, in the meantime, denote it by $c(x_2, x_1)$, which from results already obtained is clearly equal to $c(x_1, x_2)$.

Angular segments and intervals

If an angle-boundary, which is not null, lies in a separation plane S and has an element O as vertex, it divides all the rays in S having O as end, excluding the sides of the angle-boundary, into two distinct sets.

Let x_1 and x_2 be the sides of the angle-boundary (which we shall first suppose is not flat) and let A be any element of x_1 and B be any element of x_2.

Then any ray in S having O as its end which intersects AB in an element M linearly between A and B belongs to the one set; while any ray in S having O as its end, other than x_1 and x_2, which does not intersect AB in an element linearly between A and B, belongs to the other set.

It may easily be proved, by means of the analogues of Peano's axioms 13 and 14, that the property of a ray such as OM is independent of the positions of A and B in x_1 and x_2 respectively; so that the set of rays of this type is independent of the positions of A and B in the sides of the angle-boundary, and accordingly, the other set of rays is also independent of these positions.

Next, taking the case where x_1 and x_2 form a flat angle-boundary; any ray in S having O as its end which lies on one side of the complete separation line formed by x_1, x_2 and the element O, belongs to the one set; while any ray in S having O as its end which lies on the other side of this separation line belongs to the other set.

Thus in all these cases, the angle-boundary together with any one ray of a set determines that set and distinguishes it from the other set of rays having the same boundary.

In the case of a null angle-boundary, since x_1 coincides with x_2, there is no separation of the remaining rays into two sets; but, instead of this, there is only one set comprising all such rays.

We may now introduce the following:

Definitions: If x_1 and x_2 be the sides of an angle-boundary in a separation plane S having an element O as vertex, then either of the sets of rays in S having O as end separated off in the above manner by the angle-boundary (or, in the case of a null angle-boundary, the single set) will be called an *angular segment.*

The rays x_1 and x_2 will be called the *sides* of the angular segment, but are not included in it.

An angle-boundary which is not a null one together with either of

the angular segments which it separates will be called an *angular interval.*

A null angle-boundary without any angular segment will be called a *null angular interval*; while a null angle-boundary together with the single angular segment associated with it will be called a *circuit angular interval*: the segment itself being called a *circuit angular segment.*

Reverting to the case of an angle-boundary which is neither null nor flat: an angular segment or interval which contains a ray such as OM will be said to be of the *first type*; while one which does not contain a ray such as OM will be said to be of the *second type.*

An angular segment or interval of the first type will be said to be *acute*, *right* or *obtuse* according as the angle-boundary is acute, right or obtuse; but these terms do not apply to angular segments or intervals of the second type.

The two angular segments or intervals associated with a flat angle-boundary will be called flat angular segments or intervals and will also be regarded as *obtuse.*

Again, if x_1 and x_2 form an angle-boundary which is neither null nor flat, the angular segment or interval of the first type formed by x_1 and x_2' or by x_2 and x_1' will be said to be a *supplement* of the angular segment or interval of the first type formed by x_1 and x_2; but this term does not apply to angular segments or intervals of the second type.

When an angle-boundary is not null the two angular segments or intervals associated with it will be said to be *conjugate* to one another.

Also a null angular interval and a circuit angular interval will be said to be *conjugate* to one another; but there is no angular segment conjugate to a circuit angular segment.

Consider now the case of an angular segment which is not a complete circuit one, but has x_1 and x_2 as its sides and O as its vertex. We have to make a distinction between the two sides of the separation line $\bar{x}_1$ with respect to the angular segment, and this is done as follows:

If (1) all the rays of the angular segment lie on one side of $\bar{x}_1$;

or (2) all the rays in S having O as end and lying on one side of x_1 are rays of the angular segment;

then such side will be called the *positive side of $\bar{x}_1$ with respect to that angular segment* (or the corresponding angular interval), and the other will be called the negative side.

This gives a unique determination of the positive side of $\bar{x}_1$ with

respect to a given angular segment when this is not a circuit segment; but fails to do this if it be such.

It will be found however that, for our present purpose, this does not matter.

If we consider the case where the angle-boundary formed by x_1 and x_2 is neither flat nor null; then of the two conjugate angular segments into which it divides the other rays of S which have O as end, one will be of what we have called the first type and the other will be of the second type.

It is clear that the angular segment of the second type will contain the two rays x_1' and x_2'; since a separation line such as AB which intersects x_1 and x_2 cannot intersect either x_1' or x_2'.

Thus x_2 and all the rays belonging to the angular segment of the first type lie on one side of $\bar{x}_1$, and this is the positive side of $\bar{x}_1$ with respect to the angular segment of the first type.

On the other hand, all the rays in S which have O as end and which lie on the opposite side of $\bar{x}_1$ belong to the angular segment of the second type, and accordingly, this will be the positive side of $\bar{x}_1$ with respect to the angular segment of the second type.

Thus x_2 lies on the negative side of $\bar{x}_1$ with respect to the angular segment of the second type; but it lies on the positive side of $\bar{x}_1$ with respect to the angular segment of the first type.

In the case of a flat angular segment, all the rays belonging to it lie on one side of $\bar{x}_1$ and also all rays in S having O as end and lying on this same side of $\bar{x}_1$ are rays of the angular segment; so that, for a double reason in this case, this will be the positive side of $\bar{x}_1$ with respect to this flat angular segment.

The ray x_2 of course in this case actually lies in $\bar{x}_1$.

Now let $\bar{y}_1$ be the separation line passing through O and lying in S which is normal to $\bar{x}_1$ and let the half-lines into which $\bar{y}_1$ is divided by O be denoted by y_1 and y_1'; of which y_1 is taken to be the one which is on the positive side of $\bar{x}_1$ with respect to the particular angular segment we are considering.

Then the angle-boundary which x_2 and y_1 make is characterised by the function $c(x_2, y_1)$, which may be positive, zero or negative according as the angle-boundary is acute, right or obtuse.

If the angular segment we are considering be of the first type, then, as we have seen, x_2 will be on the positive side of $\bar{x}_1$, so that $c(x_2, y_1)$ will be positive.

If the angular segment be of the second type, then, as we have also

seen, x_2 will be on the negative side of $\bar{x}_1$ and so $c(x_2, y_1)$ will be negative.

If the angular segment be flat, then x_2 will lie in $\bar{x}_1$ so that $c(x_2, y_1)$ will be zero.

If the angular segment be a circuit one, then x_2 coincides with x_1, so that again we have $c(x_2, y_1)$ zero, no matter which side of $\bar{x}_1$ be taken as the positive one.

This case however differs from that of a flat angular segment in that for the latter, $c(x_2, x_1) = -1$; while for a circuit segment, $c(x_2, x_1) = +1$.

In fact it follows directly from Theorem 198, that in all cases

$$\{c(x_2, x_1)\}^2 + \{c(x_2, y_1)\}^2 = 1.$$

The angular segment, as here defined, will be characterised by the two functions $c(x_2, x_1)$ and $c(x_2, y_1)$ taken in conjunction: the one being determinate from the other except as regards sign.

It will hereafter be found convenient to denote such a pair of functions taken in conjunction by the symbol

$$c(x_2, x_1) + ic(x_2, y_1),$$

and we shall call this the De Moivre function of the angular segment.

Any angular segment similar to this will have the same De Moivre function.

Definition. Angular segments will be said to be *congruent* when their De Moivre functions are equal.

Such complex functions are regarded as equal when the corresponding component functions are separately equal each to each.

As regards angular intervals which are neither null nor circuit intervals, these will also be said to be *congruent* when their De Moivre functions are equal, but a null angular interval has the same De Moivre function as a circuit one.

They are however distinguished from one another in that a null interval has no corresponding angular segment, while a circuit interval has.

ADDITION OF ANGLES

We have now to consider a series of half-lines, $x_0, x_1, x_2, \ldots$ all having a common end O and lying in a separation plane S, along with a second series $y_0, y_1, y_2, \ldots$ also having the common end O and lying in S and such that $\bar{y}_0, \bar{y}_1, \bar{y}_2, \ldots$ are respectively normal to $\bar{x}_0, \bar{x}_1, \bar{x}_2, \ldots$.

As regards $\bar{y}_0$, it is perfectly arbitrary which of the component half lines into which it is divided by O we denote by y_0 and which by y_0',

but, a selection being made in the case of $\bar{y}_0$, we are able to assign a definite systematic nomenclature in the case of $\bar{y}_1$, $\bar{y}_2$, ...

If now we take x_0 and y_0 as standards we shall make the following conventions:

If x_1 is either identical with x_0 or else makes acute angle-boundaries with both x_0 and y_0, it will be said to lie in the first quadrant.

If x_1 is either identical with y_0 or else makes acute angle-boundaries with both y_0 and x_0', it will be said to lie in the second quadrant.

If x_1 is either identical with x_0' or else makes acute angle-boundaries with both x_0' and y_0', it will be said to lie in the third quadrant.

If x_1 is either identical with y_0' or else makes acute angle-boundaries with both y_0' and x_0, it will be said to lie in the fourth quadrant.

These cover all the possibilities which are open with regard to x_1; as is readily seen.

Omitting for the present the cases where x_1 is identical with one of the half-lines x_0, y_0, x_0', y_0', we shall consider the other possibilities in succession.

(1) Let us suppose that x_1 makes acute angle-boundaries with both x_0 and y_0 and let A be any element in x_0. Let the normal through A to $\bar{x}_1$ intersect it in M.

Then, since x_1 makes an acute angle-boundary with x_0, it follows that M must lie in x_1.

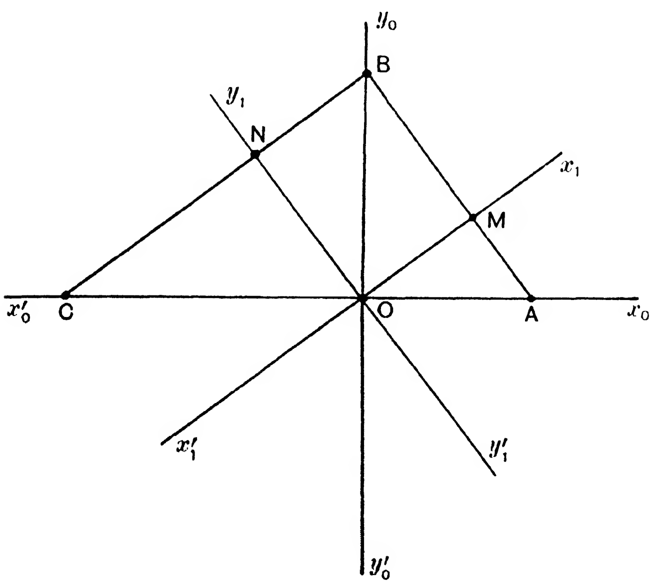


Fig. 51.

Also, since $\bar{x}_1$ does not coincide with $\bar{x}_0$, it follows that AM cannot be parallel to $\bar{y}_0$ and therefore must intersect $\bar{y}_0$ in some element B. Further, since x_1 makes an acute angle-boundary with y_0, it follows that B must lie in y_0, and, by the remarks at the end of Theorem 194, M must be linearly between A and B.

Now, since $\bar{x}_0$ is supposed to be distinct from $\bar{x}_1$, a separation line through B parallel to $\bar{x}_1$ must intersect $\bar{x}_0$ in some element C.

Further, since M is linearly between A and B we must have O linearly between A and C, so that C must lie in x_0'.

Now let the separation line $\bar{y}_1$, which passes through O and is normal to $\bar{x}_1$, intersect BC in N.

Then, since $\angle BOC$ is a right angle-boundary, it follows that N must be linearly between B and C; so that $\angle NOB$ and $\angle NOC$ are both acute.

We shall select the part of $\bar{y}_1$ which contains N as the one to be called y_1.

Thus, if x_1 makes acute angle-boundaries with x_0 and y_0, then our notation is so chosen that y_1 makes acute angle-boundaries with y_0 and x_0'.

Now, since ON is parallel to AB, it follows that:

$$ON : OC = AB : AC,$$

and, since both are acute angle-boundaries, it follows that:

$$\angle CON \equiv \angle OAB.$$

But, since BO is normal to AO, and OM is normal to AB, it follows, by Theorem 197, that:

$$\angle OAB \equiv \angle BOM.$$

Thus $$\angle CON \equiv \angle BOM.$$

Similarly we can show that:

$$\angle AOM \equiv \angle BON.$$

These last two congruences may be expressed thus:

$$\angle(y_1, x_0') \equiv \angle(x_1, y_0) \qquad \ldots\ldots(a)$$

and $$\angle(y_1, y_0) \equiv \angle(x_1, x_0) \qquad \ldots\ldots(b).$$

(2) If we carry out the above investigation making the substitution:

$$\begin{pmatrix} x_0, & y_0, & x_0', & y_0' \\ y_0, & x_0', & y_0', & x_0 \end{pmatrix}, \text{ implying that } \begin{pmatrix} \bar{x}_0, & \bar{y}_0 \\ \bar{y}_0, & \bar{x}_0 \end{pmatrix},$$

we get $$\angle(y_1, y_0') \equiv \angle(x_1, x_0') \qquad \ldots\ldots(a)$$

and $$\angle(y_1, x_0') \equiv \angle(x_1, y_0) \qquad \ldots\ldots(b).$$

(3) If instead, we make the substitution:

$$\begin{pmatrix} x_0, y_0, x_0', y_0' \\ x_0', y_0', x_0, y_0 \end{pmatrix}, \text{ implying that } \begin{pmatrix} \bar{x}_0, \bar{y}_0 \\ \bar{x}_0, \bar{y}_0 \end{pmatrix},$$

we get $\angle(y_1, x_0) \equiv \angle(x_1, y_0')$ ......(a)

and $\angle(y_1, y_0') \equiv \angle(x_1, x_0')$ ......(b).

(4) Finally, if we carry out the original investigation making the substitution:

$$\begin{pmatrix} x_0, y_0, x_0', y_0' \\ y_0', x_0, y_0, x_0' \end{pmatrix}, \text{ implying that } \begin{pmatrix} \bar{x}_0, \bar{y}_0 \\ \bar{y}_0, \bar{x}_0 \end{pmatrix},$$

we get $\angle(y_1, y_0) \equiv \angle(x_1, x_0)$ ......(a)

and $\angle(y_1, x_0) \equiv \angle(x_1, y_0')$ ......(b).

The above results may be transformed thus:

$$\begin{array}{ll} (1) \begin{cases} (a) & \text{supplement}\angle(x_1, y_0) \equiv \angle(y_1, x_0), \\ (b) & \angle(x_1, x_0) \equiv \angle(y_1, y_0), \end{cases} \\ (2) \begin{cases} (a) & \angle(x_1, x_0) \equiv \angle(y_1, y_0), \\ (b) & \text{supplement}\angle(x_1, y_0) \equiv \angle(y_1, x_0), \end{cases} \\ (3) \begin{cases} (a) & \text{supplement}\angle(x_1, y_0) \equiv \angle(y_1, x_0), \\ (b) & \angle(x_1, x_0) \equiv \angle(y_1, y_0), \end{cases} \\ (4) \begin{cases} (a) & \angle(x_1, x_0) \equiv \angle(y_1, y_0), \\ (b) & \text{supplement}\angle(x_1, y_0) \equiv \angle(y_1, x_0). \end{cases} \end{array}$$

Let us now complete the conventions of notation as regards the part of $\bar{y}_1$ which we shall call y_1 in the following manner:

If x_1 coincides with x_0, then y_1 coincides with y_0.
If x_1 coincides with y_0, then y_1 coincides with x_0'.
If x_1 coincides with x_0', then y_1 coincides with y_0'.
If x_1 coincides with y_0', then y_1 coincides with x_0.

Then, since the supplement of a right angle-boundary is a right angle-boundary, while the supplement of a null angle-boundary is a flat angle-boundary and *vice versa*, it follows at once that, with these conventions, we still have the two congruences:

$$\text{supplement}\angle(x_1, y_0) \equiv \angle(y_1, x_0)$$

and

$$\angle(x_1, x_0) \equiv \angle(y_1, y_0).$$

The conventions of notation may now be summed up as follows:

If x_1 lies in the first quadrant y_1 lies in the second quadrant.
If x_1 lies in the second quadrant y_1 lies in the third quadrant.
If x_1 lies in the third quadrant y_1 lies in the fourth quadrant.
If x_1 lies in the fourth quadrant y_1 lies in the first quadrant.

With these conventions, whatever quadrant x_1 lies in we have the above two congruences; so that, if we employ the numerical form of the c-ratios, we have in all cases:

$$-c(x_1, y_0) = c(y_1, x_0)$$

and
$$c(x_1, x_0) = c(y_1, y_0).$$

It will be observed that these results have been obtained without making use of the "addition of angles".

If now we make the substitution $\begin{pmatrix} x_0, y_0, x_1, \bar{y}_1 \\ x_1, y_1, x_2, \bar{y}_2 \end{pmatrix}$, we may carry out a similar argument fixing the part of $\bar{y}_2$ to be called y_1 and proving that:

$$-c(x_2, y_1) = c(y_2, x_1)$$

and
$$c(x_2, x_1) = c(y_2, y_1)$$

and so on in succession any number of times.

Now let A be an element in the half-line x_2 and let the length OA be denoted by r.

We may treat the pair of separation lines $\bar{x}_0, \bar{y}_0$ as a pair of Cartesian coordinate axes having O as origin and the half-lines x_0, y_0 as the positive parts of the axes.

Let ξ_0, η_0 be the coordinates of the element A in this system.

Similarly we may treat $\bar{x}_1, \bar{y}_1$ as another pair of Cartesian coordinate axes having the same origin O and the half-lines x_1, y_1 as the positive parts of the axes.

Let ξ_1, η_1 be the coordinates of the element A in this second system and let the normal through A to $\bar{x}_1$ intersect $\bar{x}_1$ in the element M.

Now the principle that the algebraic sum of the projections of the parts A_0A_1, A_1A_2, ... $A_{n-1}A_n$ of a broken line upon a given line is equal algebraically to the projection of the linear interval A_0A_n joining the extremities of the broken line upon the given line clearly holds in our geometry just as it does in ordinary Euclidean geometry: the proof being exactly analogous.

Accordingly the projection of OA upon $\bar{x}_0$ or $\bar{y}_0$ is equal to the algebraic sum of the projections of OM and MA upon the same separation lines.

Taking first the projections upon $\bar{x}_0$, we have

$$OM = \xi_1 \quad \text{and} \quad MA = \eta_1$$

while MA is co-directional with $\bar{y}_1$.

Thus
$$\begin{aligned} \xi_0 &= \xi_1 c(x_1, x_0) + \eta_1 c(y_1, x_0), \\ &= \xi_1 c(x_1, x_0) - \eta_1 c(x_1, y_0). \end{aligned}$$

But
$$\xi_0 = rc(x_2, x_0), \quad \xi_1 = rc(x_2, x_1), \quad \eta_1 = rc(x_2, y_1).$$

Thus we get

$$c(x_2, x_0) = c(x_2, x_1)\, c(x_1, x_0) - c(x_2, y_1)\, c(x_1, y_0) \quad \ldots\ldots(1).$$

Taking the projections upon $\bar{y}_0$ we get

$$\eta_0 = \xi_1 c(x_1, y_0) + \eta_1 c(y_1, y_0),$$
$$= \xi_1 c(x_1, y_0) + \eta_1 c(x_1, x_0);$$

while $$\eta_0 = rc(x_2, y_0).$$

Thus

$$c(x_2, y_0) = c(x_2, x_1)\, c(x_1, y_0) + c(x_2, y_1)\, c(x_1, x_0) \quad \ldots\ldots(2).$$

The formulae (1) and (2) may be combined into a single one by means of the symbol $i = \sqrt{-1}$; thus:

$$c(x_2, x_0) + ic(x_2, y_0)$$
$$= \{c(x_2, x_1) + ic(x_2, y_1)\}\{c(x_1, x_0) + ic(x_1, y_0)\} \quad \ldots\ldots(3).$$

It will be observed that formulae (1) and (2) are the equivalents of the addition formulae for cosine and sine respectively; while (3) is equivalent to the formula of De Moivre.

Now we have already seen that any angular segment is characterised by one definite De Moivre function and any De Moivre function is characteristic of all angular segments which are congruent to one another.

The same is true with regard to angular intervals except that a null interval and a circuit interval have the same De Moivre function.

We also remarked that it would be found convenient to denote a De Moivre function in a certain manner involving a symbol i.

We now see that by taking i to stand for $\sqrt{-1}$ we have got an interpretation for the product of the De Moivre functions of two angular segments as the De Moivre function of an angular segment bearing a simple relation to the first two taken in conjunction.

We have now to make a diversion on the exponential function.

The exponential function of an argument z or exp (z) is defined as the limit of the infinite series

$$1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots$$

and, as is well known, it has the property that

$$\exp(u) \,.\, \exp(v) = \exp(u + v)$$

for all values of u and v, real and imaginary.

If we put $z = i\theta$ we get

$$\exp(i\theta) = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots$$

and it is well known that the real and imaginary parts of this, namely:

$$1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\dots$$

and

$$\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\dots$$

called $\cos\theta$ and $\sin\theta$ respectively, are such that the sum of their squares is equal to unity.

Also it is known that corresponding to any value of $\exp(i\theta)$ there are an infinite number of values of θ of the general form $\theta_0+2n\pi$, where n is any integer positive, zero or negative.

We can thus write

$$c(x_1, x_0)+ic(x_1, y_0)=\exp(i\theta)$$

and

$$c(x_2, x_1)+ic(x_2, y_1)=\exp(i\phi)$$

and then we have

$$c(x_2, x_0)+ic(x_2, y_0)=\exp(i\overline{\theta+\phi}).$$

The quantities θ and ϕ so introduced may be either positive or negative according to the signs of the n's.

The different positive values of θ; that is to say $\theta_0+2n\pi$, where n is zero or positive and θ_0 is the smallest positive value of θ, will be called the *angles* (in natural measure) *corresponding congruently to* an angular interval whose De Moivre function is

$$c(x_1, x_0)+ic(x_1, y_0).$$

The different values of $\theta_0+2n\pi$, where n has the negative values $-1, -2, -3, \dots$ with their signs reversed, will be the angles corresponding congruently to the conjugate angular interval.

In the special case of a null interval we have the set of angles

$$0,\ 2\pi,\ 4\pi,\ 6\pi,\ \dots$$

corresponding congruently to it, and, giving negative values to n, and reversing the signs, we have the set of angles

$$2\pi,\ 4\pi,\ 6\pi,\ \dots$$

corresponding congruently to the conjugate angular interval, which is here a circuit interval.

Thus, for a null interval, θ_0 is taken as zero, while for a circuit interval θ_0 is taken as 2π, though the De Moivre function is the same for both.

The value θ_0 will then represent the magnitude of the angular interval or segment (when such exists); while $\theta_0+2n\pi$ for positive

integral values of n will represent that magnitude $+n$ times the magnitude of a circuit interval or segment.

It is to be observed that in a separation plane there exists no entity corresponding to an angular segment or interval greater than a circuit one, any more than any linear segment or interval can exist inside a circle which is greater than the diameter of the circle; although the sum of the magnitudes of several such segments or intervals may be as great as we please.

It is unnecessary to go into this subject in further detail, since it is obvious that we should merely be covering ground which is already familiar.

We have employed the above method in dealing with the measures of angles in order to avoid making use of conceptions alien to our subject: such, for instance, as the "rotation of a half-line about its end".

This mode of speech, although familiar, is appropriate to Kinematics rather than to Pure Geometry, and would be quite out of place in treating of a separation plane in which no element is either *before* or *after* any other one and in which no motion can occur.

This is particularly important in a work like the present, where we are concerned with showing how a system of geometry may be built up from certain fundamental concepts, and not merely with seeing, more or less intuitively that certain things are the case.

Once a firm basis is laid down one may proceed with the development of a subject with less circumspection, and in the present case we have reached a stage where we are safely entitled to say that the geometry of a separation plane is formally identical with that of a Euclidean plane, and, in consequence, *the geometry of a separation threefold is formally identical with the ordinary* (*Euclidean*) *geometry of three dimensions*.

We do not propose to consider the theory of areas, volumes, etc. since these are formally identical with the corresponding theories in ordinary geometry.

Theorem 199

If B and C be two distinct elements in a separation line and O be their mean, and if A be any element in an optical line a which passes through O and is normal to BC, then

$$(A, B) \equiv (A, C).$$

Since a is an optical line which is normal to the separation line BC, it follows that a and BC lie in an optical plane, say P.

If the element A should happen to coincide with O then, since BC is a separation line, the theorem obviously holds.

Suppose next that A does not coincide with O, and let d be a separation line passing through O and normal to P.

Then a and d determine an optical plane, say Q, which is completely normal to P; and, since BC and d are both separation lines and are normal to one another, it follows that they lie in a separation plane, say S.

Let D be any element of d distinct from O.

Then DO is normal to BC and so, by Theorem 186, we have

$$(D, B) \equiv (D, C).$$

But, since Q is completely normal to P, it follows that DA is normal to P and so DA is normal to both AB and AC.

Also, since D is not an element of a, and a is a generator of the optical plane Q, it follows that DA must be a separation line.

Similarly, since B and C are not elements of a, and a is a generator of the optical plane P, it follows that both BA and CA are separation lines.

Thus DA and BA must lie in a separation plane, say R_1, and DA and CA must lie in a separation plane, say R_2.

Thus, by Theorem 190, since

$$(A, D) \equiv (A, D)$$

and

$$(D, B) \equiv (D, C),$$

it follows that:

$$(A, B) \equiv (A, C)$$

as was to be proved.

Theorem 200

If O and X_0 be two distinct elements in a separation line lying in an optical plane P, then the set of all elements in P such as X where OX is a separation line and

$$(O, X) \equiv (O, X_0)$$

consists of a pair of parallel optical lines.

Let X_0' be an element in OX_0 and on the opposite side of O to that on which X_0 lies, and such that:

$$(O, X_0') \equiv (O, X_0).$$

Then X_0' is an element of the set we are considering, and it is evident that it is the only one besides X_0 lying in the separation line OX_0.

Further it is evident that O is the mean of X_0 and X_0'.

Let a, b and c be three generators of the optical plane P passing through X_0, X_0' and O respectively.

Let X_1 be any element in a distinct from X_0, and let OX_1 intersect b in X_1'.

Further, let c intersect $X_0'X_1$ in the element M.

Then OX_1 and $X_0'X_1$ are both separation lines, since they have each got elements in two distinct generators of the optical plane.

Now, since c must be parallel to a, and since O is the mean of X_0 and X_0', it follows, by Theorem 94, that M is the mean of X_1 and X_0'.

But, since OM is an optical line and $X_0'X_1$ is a separation line in the same optical plane P with it, it follows that OM is normal to $X_0'X_1$.

Thus, by Theorem 199, we must have

$$(O, X_1) \equiv (O, X_0').$$

But, since $$(O, X_0') \equiv (O, X_0),$$

it follows that: $$(O, X_1) \equiv (O, X_0).$$

Similarly $$(O, X_1') \equiv (O, X_0'),$$

and so $$(O, X_1') \equiv (O, X_0).$$

Thus X_1 and X_1' are evidently elements of the set we are considering, and are clearly the only ones lying in the separation line OX_1.

Similarly any other separation line passing through O and lying in P will intersect a and b in elements belonging to the set considered, and these will be the only ones lying in that separation line.

Thus the parallel optical lines a and b together constitute the set of elements in P, such as X, where OX is a separation line and

$$(O, X) \equiv (O, X_0),$$

and so the theorem is proved.

Remarks

Certain interesting results follow directly from the last theorem.

Thus if we consider any triangle in an optical plane whose corners are A, B and C, then not more than one of the general lines AB, BC, CA can be an optical line, since no two optical lines in an optical plane can intersect.

If BC be an optical line, then AB and CA must be separation lines, and from the last theorem it follows that:

$$(A, B) \equiv (A, C).$$

If, on the other hand, neither AB, BC nor CA be an optical line, they must all be separation lines.

In this case, let a, b and c be generators of the optical plane passing

through A, B and C respectively, and intersecting BC, CA and AB in A', B' and C' respectively.

Then, since neither AB, BC nor CA are optical lines, it follows that neither A', B' nor C' can coincide with a corner of the triangle.

Thus we must either have

(1) A' linearly between B and C,

or (2) C linearly between A' and B,

or (3) B linearly between C and A'.

In the first case, we shall also have

A linearly between B' and C,

and A linearly between B and C'.

In the second case, we shall also have

C linearly between A and B',

and C' linearly between A and B.

In the third case, we shall also have

B linearly between C' and A,

and B' linearly between C and A.

Thus in all cases one of the three elements A', B', C', and only one, lies linearly between a pair of the corners A, B, C.

Now let us consider the case, for instance, where A' is linearly between B and C.

It follows directly from the last theorem that:

$$(B, A) \equiv (B, A'),$$

and

$$(C, A) \equiv (C, A').$$

This remarkable result may be expressed as follows:

If all three sides of a triangle in an optical plane be separation segments, then the sum of the lengths of a certain two of the sides is equal to that of the third side.

Again, if a and b be a pair of neutral-parallel optical lines and if A_1 and A_2 be any elements in a, while B_1 and B_2 are any elements in b, we have

$$(A_1, B_1) \equiv (A_1, B_2),$$

and

$$(B_2, A_1) \equiv (B_2, A_2).$$

Thus we see that we must have

$$(A_1, B_1) \equiv (A_2, B_2).$$

It will be observed that, in the case of an optical plane, a pair of parallel optical lines is the analogue of a circle, in so far as any analogue exists.

Again, if W be an optical threefold and O be any element in it, while c is the generator of W which passes through O, then any general plane in W which contains c is an optical plane, while any one which passes through O, but does not contain c, is a separation plane.

If then S be any separation plane lying in W and passing through O and X_0 be any element in it distinct from O, the set of elements in S, such as X, where

$$(O, X) \equiv (O, X_0),$$

constitutes a separation circle.

If through each element of the separation circle a generator of W be taken, then any element X on any such generator will also satisfy the relation

$$(O, X) \equiv (O, X_0).$$

Further, it is clear that no other element of W does satisfy it.

The set of elements thus obtained lie on a sort of cylinder which, in the case of an optical threefold, takes the place of a sphere.

We shall call this an *optical circular cylinder*.

THEOREM 201

If A_1, B_1, C_1 be the corners of a triangle in an inertia plane P_1 and A_2, B_2, C_2 be the corners of a triangle in an inertia plane P_2, and if further B_1C_1 be a separation line which is normal to the inertia line A_1C_1, while B_2C_2 is a separation line which is normal to the inertia line A_2C_2, then:

(1) *If* $(C_1, A_1) \equiv (C_2, A_2)$

and $(C_1, B_1) \equiv (C_2, B_2)$,

we shall either have $(A_1, B_1) \equiv (A_2, B_2)$,

or else both A_1B_1 and A_2B_2 will be optical lines.

(2) *If* $(A_1, C_1) \equiv (C_2, A_2)$

and $(C_1, B_1) \equiv (C_2, B_2)$,

we shall either have $(A_1, B_1) \equiv (B_2, A_2)$,

or else both A_1B_1 and B_2A_2 will be optical lines.

Consider first part (1) of the theorem.

Since $(C_1, A_1) \equiv (C_2, A_2)$, and since these are inertia pairs, we must have either A_1 *before* C_1 and A_2 *before* C_2, or else have A_1 *after* C_1 and A_2 *after* C_2.

We shall only consider the case where A_1 is *before* C_1 and A_2 *before* C_2, since the other case is quite analogous.

If A_1B_1 were an optical line we should have (C_1, A_1) a before-conjugate to (C_1, B_1) and if A_2' were an element in C_2A_2, such that

(C_2, A_2') were a before-conjugate to (C_2, B_2), then it would follow, by Theorem 179, that we must have

$$(C_1, A_1) \equiv (C_2, A_2'),$$

and so we should have

$$(C_2, A_2) \equiv (C_2, A_2').$$

Thus A_2' would be identical with A_2, and so $A_2 B_2$ would also be an optical line.

We are not, however, at liberty to assert in this case that:

$$(A_1, B_1) \equiv (A_2, B_2),$$

but only that they are both optical pairs.

We shall suppose next that $A_1 B_1$ is not an optical line, and that accordingly $A_2 B_2$ is not an optical line.

Let D_1 and D_2 be elements in $C_1 A_1$ and $C_2 A_2$ respectively, such that (C_1, D_1) is a before-conjugate to (C_1, B_1), and (C_2, D_2) is a before-conjugate to (C_2, B_2).

Then, by Theorem 179, we must have

$$(C_1, D_1) \equiv (C_2, D_2).$$

Now two cases occur; we may have

(1) A_1 linearly between D_1 and C_1,

or (2) D_1 linearly between A_1 and C_1.

In the first case, since we also have

$$(C_1, A_1) \equiv (C_2, A_2),$$

it follows that we must also have A_2 linearly between D_2 and C_2, as was shown in the remarks at the end of Theorem 184.

Similarly, in the second case we must also have D_2 linearly between A_2 and C_2.

Again, in the first case we have D_1 *before* C_1, and must therefore have A_1 *after* D_1 and *before* C_1.

But A_1 could not be *before* B_1, for then A_1 would require to lie in the optical line $D_1 B_1$, which we know is not the case.

Further, A_1 could not be *after* B_1, for then, since C_1 is *after* A_1, we should have C_1 *after* B_1 contrary to the hypothesis that $B_1 C_1$ is a separation line.

It follows that in case (1) $A_1 B_1$ is a separation line, and similarly $A_2 B_2$ is a separation line.

In case (2), on the other hand, we must have A_1 *before* D_1 and so, since D_1 is *before* B_1, we must have A_1 *before* B_1.

Thus, since $D_1 A_1$ is an inertia line, and since D_1 is the only element

common to it and the β sub-set of B_1, it follows in this case that $A_1 B_1$ is an inertia line, and similarly $A_2 B_2$ is an inertia line.

We shall consider cases (1) and (2) separately.

Case (1).

We have here got $A_1 B_1$ and $A_2 B_2$, both separation lines.

Now let W_1 and W_2 be inertia threefolds containing P_1 and P_2 respectively, and let S_1 and S_2 be the separation planes in W_1 and W_2 which pass through C_1 and C_2, and are normal to the inertia lines $A_1 C_1$ and $A_2 C_2$ respectively.

Then, since $B_1 C_1$ is normal to $A_1 C_1$, it follows that $B_1 C_1$ must lie in S_1 and similarly $B_2 C_2$ must lie in S_2.

Now, since $A_1 B_1$ is a separation line, there is an inertia plane which passes through A_1, lies in W_1 and is normal to $A_1 B_1$.

This inertia plane contains two optical lines which pass through A_1 and must be normal to $A_1 B_1$ and which must intersect S_1, since S_1 is a separation plane in the same inertia threefold along with these optical lines.

Let one of these optical lines intersect S_1 in the element E_1.

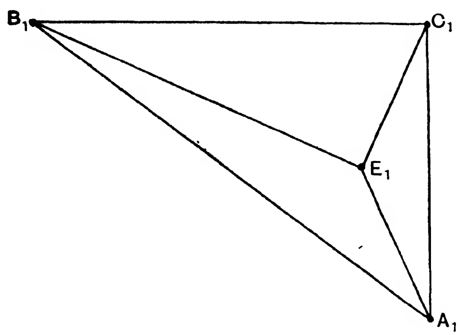


Fig. 52.

Similarly we can show that there are two optical lines passing through A_2 and lying in W_2, and which are normal to $A_2 B_2$.

These optical lines may be shown in a similar manner to intersect S_2, and we shall suppose that one of them intersects S_2 in the element E_2.

Now, since the optical line $A_1 E_1$ is normal to the separation line $A_1 B_1$, it follows that $A_1 E_1$ and $A_1 B_1$ lie in an optical plane.

Similarly $A_2 E_2$ and $A_2 B_2$ lie in an optical plane.

But, since an optical line in an optical plane is normal to every general line in the optical plane, it follows that $A_1 E_1$ is normal to $E_1 B_1$, and similarly $A_2 E_2$ is normal to $E_2 B_2$.

Again, since $E_1 B_1$ lies in S_1 and since S_1 is normal to $A_1 C_1$, it follows that $A_1 C_1$ is normal to $E_1 B_1$.

Thus $E_1 B_1$ is normal to the two intersecting general lines $A_1 E_1$ and $A_1 C_1$, and is therefore normal to the general plane containing them.

It follows that $E_1 B_1$ is normal to $E_1 C_1$, and similarly $E_2 B_2$ is normal to $E_2 C_2$.

Again, since S_1 is normal to $A_1 C_1$, it follows that $E_1 C_1$ is normal to $A_1 C_1$, and similarly it follows that $E_2 C_2$ is normal to $A_2 C_2$.

Thus, since $A_1 C_1$ and $A_2 C_2$ are inertia lines while $A_1 E_1$ and $A_2 E_2$ are optical lines, it follows that (C_1, E_1) and (C_2, E_2) are after-conjugates to (C_1, A_1) and (C_2, A_2) respectively.

But, since $$(C_1, A_1) \equiv (C_2, A_2),$$
it follows by Theorem 179 that:
$$(C_1, E_1) \equiv (C_2, E_2).$$

Thus C_1, B_1, E_1 are the corners of a triangle in the separation plane S_1 and C_2, B_2, E_2 are the corners of a triangle in the separation plane S_2, while further
$$(E_1, C_1) \equiv (E_2, C_2),$$
$$(C_1, B_1) \equiv (C_2, B_2),$$
and also $B_1 E_1$ is normal to $C_1 E_1$ and $B_2 E_2$ is normal to $C_2 E_2$, and so, by Theorem 190,
$$(E_1, B_1) \equiv (E_2, B_2).$$

But since $E_1 B_1$ and $A_1 B_1$ are separation lines lying in an optical plane, of which $A_1 E_1$ is a generator, it follows from the remarks at the end of Theorem 200 that:
$$(E_1, B_1) \equiv (A_1, B_1).$$
Similarly $$(E_2, B_2) \equiv (A_2, B_2).$$
Thus we get finally $$(A_1, B_1) \equiv (A_2, B_2),$$
and so the theorem is proved in case (1).

Case (2).

We have here got $A_1 B_1$ and $A_2 B_2$, both inertia lines.

As before, let W_1 and W_2 be inertia threefolds containing P_1 and P_2 respectively, and let S_1 and S_2 be the separation planes in W_1 and W_2 which pass through C_1 and C_2 and are normal to the inertia lines $A_1 C_1$ and $A_2 C_2$ respectively.

Then, as in the first case, $B_1 C_1$ lies in S_1 and $B_2 C_2$ lies in S_2.

Let b_1 be the separation line in S_1 which passes through B_1 and is normal to $B_1 C_1$, and similarly let b_2 be the separation line in S_2 which passes through B_2 and is normal to $B_2 C_2$.

Then, since $A_1 B_1$ is an inertia line, it follows that $A_1 B_1$ and b_1 lie in an inertia plane, say Q_1, and similarly $A_2 B_2$ and b_2 lie in an inertia plane, say Q_2.

Let one of the generators of Q_1 which pass through A_1 intersect b_1 in the element F_1, and let one of the generators of Q_2 which pass through A_2 intersect b_2 in the element F_2.

Now, since $A_1 C_1$ is an inertia line, it follows that $A_1 C_1$ and $A_1 F_1$ determine an inertia plane, and similarly $A_2 C_2$ and $A_2 F_2$ determine an inertia plane.

Since $C_1 F_1$ lies in S_1, it must be normal to $A_1 C_1$, and since $C_2 F_2$ lies in S_2, it must be normal to $A_2 C_2$.

Thus, since $A_1 F_1$ and $A_2 F_2$ are optical lines, it follows that (C_1, F_1) (C_2, F_2) are after-conjugates to (C_1, A_1) and (C_2, A_2) respectively, and so, since

$$(C_1, A_1) \equiv (C_2, A_2),$$

it follows, by Theorem 179, that:

$$(C_1, F_1) \equiv (C_2, F_2).$$

But now C_1, F_1, B_1 are the corners of a triangle in the separation

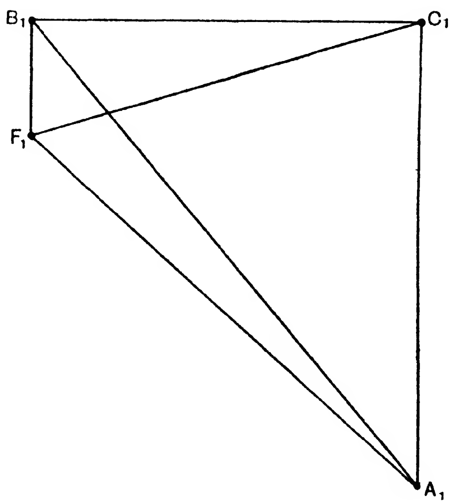


Fig. 53.

plane S_1 and C_2, F_2, B_2 are the corners of a triangle in the separation plane S_2, while further

$$(C_1, B_1) \equiv (C_2, B_2),$$
$$(C_1, F_1) \equiv (C_2, F_2),$$

and also $F_1 B_1$ is normal to $C_1 B_1$ and $F_2 B_2$ is normal to $C_2 B_2$ and so, by Theorem 190,

$$(B_1, F_1) \equiv (B_2, F_2).$$

Since $F_1 B_1$ lies in S_1, it is normal to $A_1 C_1$, and by hypothesis it is also normal to $B_1 C_1$ and so, since $A_1 C_1$ and $B_1 C_1$ are intersecting general lines in P_1, it follows that $F_1 B_1$ is normal to P_1.

Thus $F_1 B_1$ must be normal to $A_1 B_1$ and similarly $F_2 B_2$ must be normal to $A_2 B_2$.

But $A_1 F_1$ and $A_2 F_2$ are optical lines while $A_1 B_1$ and $A_2 B_2$ are inertia lines and so (B_1, F_1) and (B_2, F_2) are after-conjugates to (B_1, A_1) and (B_2, A_2) respectively.

Thus since $$(B_1, F_1) \equiv (B_2, F_2),$$

it follows, by Theorem 179, that:

$$(B_1, A_1) \equiv (B_2, A_2),$$

that is $$(A_1, B_1) \equiv (A_2, B_2),$$

as was to be proved.

Consider now part (2) of the theorem.

Since $(A_1, C_1) \equiv (C_2, A_2)$ and since these are inertia pairs we must either have A_1 *before* C_1 and A_2 *after* C_2 or else have A_1 *after* C_1 and A_2 *before* C_2.

There is then no difficulty in showing that:

$$(A_1, B_1) \equiv (B_2, A_2),$$

provided that $A_1 B_1$ be not an optical line.

The proof is quite analogous to that of the first part of the theorem except that we make use of the result given at the end of Theorem 179 in place of Theorem 179 itself.

It is also evident that if $A_1 B_1$ be an optical line, then $B_2 A_2$ must also be an optical line.

Thus both parts of the theorem hold.

It will be observed that the two parts of Theorem 201 are the analogue for inertia planes of Theorem 189.

Analogues of the theorem of Pythagoras in inertia and optical planes

In Theorem 198 we proved that the equivalent of the theorem of Pythagoras holds in a separation plane and we now propose to make use of the constructions of the two cases of Theorem 201 in order to obtain the analogue for the case of an inertia plane.

It is only necessary to consider the construction in connexion with one of the two triangles: say that whose corners are A_1, B_1, C_1.

In case (1) B_1C_1 is a separation line, A_1C_1 is an inertia line normal to B_1C_1, and A_1B_1 is a separation line.

But now we obtained a triangle whose corners were B_1, C_1 and E_1 which lay in the separation plane S_1 and such that E_1B_1 was normal to E_1C_1 and in which accordingly we must have the segment relation:

$$(B_1C_1)^2 = (E_1B_1)^2 + (E_1C_1)^2.$$

This triangle was related to the one whose corners are A_1, B_1, C_1 in such a way that

$$(E_1, B_1) \equiv (B_1, A_1),$$

while (C_1, E_1) was a before- or after-conjugate to (C_1, A_1).

Thus taking segments instead of pairs we get

$$(B_1C_1)^2 = (B_1A_1)^2 + (\text{conjugate } C_1A_1)^2.$$

Thus the analogue of the theorem of Pythagoras is in this case

$$(B_1A_1)^2 = (B_1C_1)^2 - (\text{conjugate } C_1A_1)^2 \qquad \text{......(i).}$$

Again if we consider case (2) we have B_1C_1 is a separation line, A_1C_1 is an inertia line normal to B_1C_1, and A_1B_1 is also an inertia line.

In this case we obtained a triangle whose corners were C_1, B_1 and F_1 which lay in the separation plane S_1 and such that B_1F_1 was normal to B_1C_1.

Thus we must have the segment relation:

$$(C_1F_1)^2 = (B_1C_1)^2 + (B_1F_1)^2.$$

This triangle was related to the one whose corners are A_1, B_1, C_1 in such a way that (C_1, F_1) was a before- or after-conjugate to (C_1, A_1) while (B_1, F_1) was a before- or after-conjugate to (B_1, A_1), and so, taking segments instead of pairs, we get

$$(\text{conjugate } C_1A_1)^2 = (B_1C_1)^2 + (\text{conjugate } B_1A_1)^2.$$

Thus the analogue of the theorem of Pythagoras is in this case

$$-(\text{conjugate } B_1A_1)^2 = (B_1C_1)^2 - (\text{conjugate } C_1A_1)^2 \quad \text{...(ii).}$$

In the case where A_1B_1 is an optical line we obviously have

$$0 = (B_1C_1)^2 - (\text{conjugate } C_1A_1)^2 \qquad \text{......(iii).}$$

Thus (i), (ii) and (iii) constitute the complete analogue of the Pythagorean theorem in an inertia plane provided that A_1, B_1 and C_1 form the corners of a triangle.*

If we consider a triangle whose corners are A_1, B_1, C_1 and which lies in an optical plane, then if B_1C_1 be a separation line and A_1C_1 be normal

* Cf. footnote, p. 369.

to B_1C_1 we know that A_1C_1 must be an optical line, while A_1B_1 must be another separation line.

Now we have shown that:

$$(B_1, A_1) \equiv (B_1, C_1),$$

and so taking segments instead of pairs we see that:

$$(B_1A_1)^2 = (B_1C_1)^2 \qquad \text{......(iv).}$$

This is the analogue of the Pythagorean theorem in an optical plane provided that A_1, B_1 and C_1 form the corners of a triangle.*

Considering now equations (i), (ii), (iii) and (iv) we observe that *the modifications which take place in the theorem of Pythagoras are such that when any side of the triangle becomes an inertia segment the corresponding square is replaced by the negative square of the conjugate of this inertia segment, while if any side becomes an optical segment, the corresponding square is replaced by zero.*

If we consider equation (i) we see that:

$$(B_1A_1)^2 < (B_1C_1)^2$$

and accordingly $\qquad B_1A_1 < B_1C_1.$

Again, if we consider equation (ii) we see that:

$$(\text{conjugate } B_1A_1)^2 < (\text{conjugate } C_1A_1)^2$$

and accordingly $\qquad B_1A_1 < C_1A_1.$

Thus, *provided that the hypotenuse is not an optical line, its length is less than the length of that side which is of the same kind as itself.*

We shall now make use of this result in order to prove another important theorem concerning triangles in an inertia plane.

Let A, B, C be the corners of a triangle in an inertia plane P, and let AB, BC and CA be all separation lines or all inertia lines.

It is easy to see that triangles of both these kinds exist, although, as Theorem 14 shows, it is not possible for AB, BC and CA to be all optical lines.

Let a_1, b_1, c_1 be generators of P of one set, which pass through A, B, C respectively and intersect BC, CA, AB in A_1, B_1, C_1 respectively.

* There is a limiting case of analogue to the theorem of Pythagoras in which the elements A_1, B_1 and C_1 do not form the corners of a triangle but all lie in one optical line. It may be stated in the following form:

If A_1, B_1 and C_1 be three distinct elements and if B_1C_1 and C_1A_1 be segments of optical lines which are normal to one another, then B_1A_1 is also a segment of an optical line. This follows directly since B_1C_1 and C_1A_1 being normal to one another, must be segments of the same optical line, as must also B_1A_1.

Then we may show by a method similar to that employed in the remarks at the end of Theorem 200, that one and only one of the elements A_1, B_1, C_1 is linearly between a pair of the corners A, B, C.

Similarly we may show that if a_2, b_2, c_2 be generators of P of the opposite set passing through the elements A, B, C respectively and intersecting BC, CA, AB in A_2, B_2, C_2 respectively, then one and only one of the elements A_2, B_2, C_2 is linearly between a pair of the corners A, B, C.

Now consider the case, for instance, where A_1 is linearly between B and C, and suppose first that AB, BC, CA are all separation lines.

Then B cannot be linearly between A_1 and A_2 for then, by Theorem 73 (a) and (b), AB would require to be an inertia line, contrary to hypothesis.

Similarly C cannot be linearly between A_1 and A_2.

Thus, since obviously A_2 cannot be identical with either B or C, it follows that A_2 must be also linearly between B and C.

Now let O be the mean of A_1 and A_2.

Then O is linearly between A_1 and A_2, and therefore clearly it must lie linearly between B and C.

But now A_1, A, A_2 are three corners of an optical parallelogram of which O is the centre, and so AO must be normal to A_1A_2: that is to BC.

Again, if instead of AB, BC, CA being all separation lines they are all inertia lines, a similar result holds.

Let us take the case where A_1 is linearly between B and C.

Then clearly B cannot be linearly between A_1 and A_2, for then AB would require to be a separation line, and, for a similar reason, C cannot be linearly between A_1 and A_2.

Thus, since A_2 cannot coincide with either B or C, it follows that A_2 must also be linearly between B and C.

As in the former case, if O be the mean of A_1 and A_2, then O must be linearly between B and C, and AO must be normal to BC.

Making use of the result proved above we see that, whether the sides be all separation lines, or all inertia lines we have

segment BA is less than segment BO,

and segment AC is less than segment OC.

Thus it follows that the sum of the lengths of the segments BA and AC is less than that of the segment BC.

Now we know that in a separation plane the sum of the lengths of any two sides of a triangle is greater than that of the third, and thus,

remembering what was proved at the end of Theorem 199, we have the following interesting results:

If A, B, C be the corners of a general triangle all whose sides are segments of one kind, then:

(1) *If the triangle lies in a separation plane, the sum of the lengths of any two sides is greater than that of the third side.*

(2) *If the triangle lies in an optical plane, the sum of the lengths of a certain two sides is equal to that of the third side.*

(3) *If the triangle lies in an inertia plane, the sum of the lengths of a certain two sides is less than that of the third side.*

THEOREM 202

If A, B, C be three distinct elements in an inertia plane P which do not all lie in one general line and if

$$(A, B) \equiv (A, C),$$

or if

$$(B, A) \equiv (A, C),$$

then BC cannot be an optical line.

Since the only congruence of optical pairs is co-directional, it is evident that neither AB nor AC can be optical lines and must therefore be either inertia or separation lines.

Consider first the case where they are inertia lines and

$$(A, B) \equiv (A, C).$$

It is evident that we must either have A *before* both B and C or *after* both B and C.

Suppose A is *before* both B and C and let a be a separation line passing through A and normal to the inertia plane containing AB and AC.

Let D be an element in a such that (A, D) is a before-conjugate to (A, B).

Then (A, D) will also be a before-conjugate to (A, C) since

$$(A, B) \equiv (A, C).$$

Thus DB and DC will both be optical lines, and so BC cannot be an optical line.*

If A be *after* both B and C the result follows in a similar manner.

Next consider the case where AB and AC are inertia lines but where

$$(B, A) \equiv (A, C).$$

* It is also to be noted that B is neither *before* nor *after* C in this case.

We must then either have A *after* B and C *after* A or else A *before* B and C *before* A.

In either case it is evident that BC could not be an optical line, for otherwise A would be *after* one element of it and *before* another and yet not lie in the optical line; which we know to be impossible.

Consider next the case where AB and AC are separation lines and where accordingly

$$(A, B) \equiv (A, C)$$

implies

$$(B, A) \equiv (A, C),$$

and conversely.

Now we know that there is one single optical parallelogram in P having A as centre and B as one of its corners.

Suppose, if possible, that BC is an optical line which we shall denote shortly by b.

Then b would be one of the side lines of this optical parallelogram, and we shall denote the opposite side line by b'.

Let B' be the corner opposite to B and let D and D' be the remaining two corners: D lying in b and D' lying in b'.

Let CA intersect b' in the element C' and let optical lines passing through C and C' respectively and parallel to BD' intersect b' and b in E' and E respectively.

Then E', C, E, C' would form the corners of an optical parallelogram having also b and b' as a pair of opposite side lines.

Thus, since the diagonal line CC' passes through A, it follows, by Theorem 64, that these two optical parallelograms would have a common centre A.

But now either (A, D) or (A, D') would be an after-conjugate to (A, B) while (A, E) or (A, E') would be an after-conjugate to (A, C) and DE and $D'E'$ would both be optical lines.

Thus by the first case of the theorem it is impossible that we should have

$$(A, D) \equiv (A, E),$$

or

$$(A, D') \equiv (A, E').$$

If however we had

$$(A, B) \equiv (A, C),$$

these other congruences would require to hold and so it is impossible to have BC an optical line if

$$(A, B) \equiv (A, C).$$

Thus the theorem holds in all cases.

It is important to note that while this result holds for an inertia plane, it does not, as we have already shown, hold for an optical plane.

Thus since an optical line can only lie in an inertia or optical plane, it follows that:

If B and C be two distinct elements in an optical line while A is an element which does not lie in BC, then if

$$(A, B) \equiv (A, C)$$

the elements A, B, C must lie in an optical plane.

In the remaining theorems to be considered dealing with triangles in inertia planes, we propose to treat of *equality of segments* rather than of *congruence of pairs*; as was done in connexion with triangles in separation planes.

In the latter case it is a matter of indifference whether we consider the congruence of pairs or the equality of segments; since the two subjects run parallel; but in inertia planes things are somewhat different.

In order to deal with the congruence of pairs in inertia planes, it is often necessary to make the enunciation of theorems very complicated in order to cover the various possibilities which occur as regards *before* and *after* relations of the pairs in inertia or optical lines.

If we deal with the equality of segments, on the other hand, it is generally easy, in any particular case, to express the result in the notation of pairs if so required.

Theorem 203

If A, B and C be three distinct elements which lie in an inertia plane P, but do not all lie in one general line and if O be the mean of B and C, then if

$$AB = AC,$$

or if AB and AC be both optical lines, we must have AO normal to BC.

The conditions of this theorem could not be satisfied if BC were an optical line for, in the first place, since A, B and C lie in an inertia plane, it follows from the last theorem that we could not have $AB = AC$.

In the second place, Theorem 14 shows that, if BC were an optical line, we could not have AB and AC also optical lines.

It follows that BC must either be a separation line or an inertia line.

In either of these cases, if AB and AC were both optical lines, then A, B and C would be three corners of an optical parallelogram of

which O would be the centre and so AO would be normal to BC by definition.

Next suppose that BC is a separation or inertia line and let the normal to BC through A intersect BC in O'.

Then in all cases, considering the two triangles $AO'B$ and $AO'C$, we have

$$AB = AC$$

and AO' common, so that, applying the analogue of the theorem of Pythagoras, we get either

$$(BO')^2 = (CO')^2,$$

or

$$(\text{conjugate } BO')^2 = (\text{conjugate } CO')^2.$$

Thus we have

$$BO' = CO',$$

so that O' must be the mean of B and C and therefore must be identical with O.

Thus since, by hypothesis AO' is normal to BC, we must have AO normal to BC, as was to be proved.

We have incidentally proved that:

If A, B and C be three distinct elements which lie in an inertia plane P, but do not all lie in one general line and if O be an element in BC such that AO is normal to BC, then if

$$AB = AC,$$

or if AB and AC be both optical lines, the element O must be the mean of B and C.

THEOREM 204

If A_1, B_1, C_1 be the corners of a triangle in an inertia plane P_1, and A_2, B_2, C_2 be the corners of a triangle in an inertia plane P_2, and if further B_1C_1 is normal to A_1C_1 and B_2C_2 is normal to A_2C_2, then if the segments

$$C_1A_1 = C_2A_2,$$

and

$$A_1B_1 = A_2B_2,$$

or, alternately to the latter equality, if A_1B_1 and A_2B_2 be both optical lines, we shall also have

$$C_1B_1 = C_2B_2.$$

It will be observed that, except that we have taken segments instead of pairs, this is the analogue of Theorem 190 and may be proved in a similar manner, using Theorem 201 in place of Theorem 189 and Theorem 203 in place of Theorem 187.

It should be noticed that neither C_1A_1 nor C_1B_1 can be optical lines for, since they are normal, they would then coincide and A_1, B_1, C_1 could not be corners of a triangle. Similarly, neither C_2A_2 nor C_2B_2 can be optical lines.

This point is required in applying Theorem 203 but otherwise the proof is quite analogous.

THEOREM 205

If A_1, B_1, C_1 be the corners of a triangle in an inertia plane P_1, and A_2, B_2, C_2 be the corners of a triangle in an inertia plane P_2, and if A_1C_1 be a separation line which is normal to the inertia line B_1C_1, then if the segments

$$A_1C_1 = A_2C_2,$$
$$B_1C_1 = B_2C_2,$$

and if
$$A_1B_1 = A_2B_2,$$

or, alternately to the last equality, if A_1B_1 and A_2B_2 be both optical lines, we must also have A_2C_2 normal to B_2C_2.

From the equalities it follows that, since A_1C_1 is a separation line, A_2C_2 must be a separation line, and, since B_1C_1 is an inertia line, B_2C_2 must be an inertia line.

Thus any general line normal to B_2C_2 must be a separation line.

Except that we have taken segments instead of pairs, this is the analogue of Theorem 191 and may be proved in a similar manner with some slight modifications.

Using the same notation employed in Theorem 191, the only point to be noticed in the case where A_1B_1 and A_2B_2 are not optical lines is that, since

$$A_2C_2 = A_2'C_2,$$

the hypothetical general line A_2A_2' could not be an optical line, by Theorem 202, and so the normal to it through O could not coincide with itself.

Apart from this the proof is similar to that of Theorem 191 using Theorem 201 in place of Theorem 189 and Theorem 203 in place of Theorem 187.

In the case where A_1B_1 and A_2B_2 are both optical lines the following point is to be noted: If A_2' lies in A_2B_2 it must coincide with A_2: not because we are entitled to assert the equality of A_2B_2 and A_2',B_2, but because, if it did not do so, we should have

$$A_2C_2 = A_2'C_2$$

and A_2A_2' an optical line which we know is impossible.

Apart from this the proof is similar to that in the case where $A_1 B_1$ and $A_2 B_2$ are not optical lines.

Remarks

This theorem may be used to prove a converse to the analogue of the theorem of Pythagoras in an inertia plane.

Let A_1, B_1 and C_1 be the corners of a triangle in an inertia plane and let $B_1 C_1$ be a segment of a separation line while $C_1 A_1$ is a segment of an inertia line and suppose that one of the three following conditions holds:

(i) $B_1 A_1$ is a segment of a separation line and

$$(B_1 A_1)^2 = (B_1 C_1)^2 - (\text{conjugate } C_1 A_1)^2;$$

(ii) $B_1 A_1$ is a segment of an inertia line and

$$-(\text{conjugate } B_1 A_1)^2 = (B_1 C_1)^2 - (\text{conjugate } C_1 A_1)^2;$$

(iii) $B_1 A_1$ is a segment of an optical line and

$$O = (B_1 C_1)^2 - (\text{conjugate } C_1 A_1)^2;$$

we shall prove that $A_1 C_1$ is normal to $B_1 C_1$.

Consider a second triangle in an inertia plane and let its corners be A_2, B_2, C_2.

Let $B_2 C_2$ be a separation line and let $A_2 C_2$ be an inertia line which is normal to $B_2 C_2$ and let the segments

$$B_2 C_2 = B_1 C_1$$

and

$$C_2 A_2 = C_1 A_1.$$

Then we shall have one of the three conditions:

(1) $B_2 A_2$ is a segment of a separation line and

$$(B_2 A_2)^2 = (B_2 C_2)^2 - (\text{conjugate } C_2 A_2)^2;$$

(2) $B_2 A_2$ is a segment of an inertia line and

$$-(\text{conjugate } B_2 A_2)^2 = (B_2 C_2)^2 - (\text{conjugate } C_2 A_2)^2;$$

(3) $B_2 A_2$ is a segment of an optical line and

$$O = (B_2 C_2)^2 - (\text{conjugate } C_2 A_2)^2.$$

Thus we must either have

$$B_2 A_2 = B_1 A_1,$$

or else both $B_2 A_2$ and $B_1 A_1$ must be segments of optical lines.

Thus the conditions of Theorem 205 hold between the two triangles and so $A_1 C_1$ must be normal to $B_1 C_1$.

It is obvious that the conditions (i), (ii) or (iii) could only hold in an inertia plane since inertia segments are involved in each of them.

Consider now a triangle whose corners are A_1, B_1 and C_1 and suppose the condition holds:

(iv) A_1C_1 is a segment of an optical line and

$$(B_1A_1)^2=(B_1C_1)^2.$$

Then $$B_1A_1=B_1C_1,$$

and so, as was shown at the end of Theorem 202, the triangle must lie in an optical plane and, since A_1C_1 is a segment of an optical line, we must have A_1C_1 normal to B_1C_1 and also normal to B_1A_1.

Consider now a triangle whose corners are A_1, B_1 and C_1 and in which the segment relation holds:

(v) $$(B_1A_1)^2=(B_1C_1)^2+(C_1A_1)^2.$$

It is obvious that in this triangle the sides are all segments of one kind and the sum of the lengths of any two sides is greater than that of the third side, and accordingly it can only lie in a separation plane and the sides must be segments of separation lines.

By a method similar to that which we employed in dealing with conditions (i), (ii) and (iii), but using Theorem 191 in place of Theorem 205, we can prove that A_1C_1 must be normal to B_1C_1.

There is a limiting case of analogue to the theorem of Pythagoras in which the segments do not form a triangle and which was mentioned in the footnote on p. 369.

The converse of this also holds and may be stated as follows:

(vi) If A_1, B_1 and C_1 be three distinct elements and if B_1C_1, C_1A_1 and B_1A_1 be all segments of optical lines, then A_1C_1 must be normal to B_1C_1.

For, by Theorem 14, A_1, B_1 and C_1 cannot lie in pairs in three distinct optical lines and must therefore all lie in one optical line.

Thus A_1C_1 must be normal to B_1C_1.

We have thus got the complete converse to the various forms of analogue to the theorem of Pythagoras.

Theorem 206

If A_1, B_1, C_1 be the corners of a triangle in an inertia plane P_1, and A_2, B_2, C_2 be the corners of a triangle in an inertia plane P_2: the triangles being such that no side of either is an optical line; and if the segments

$$A_1B_1=A_2B_2,$$
$$A_1C_1=A_2C_2,$$
$$B_1C_1=B_2C_2,$$

while N_1 is an element in B_1C_1 such that A_1N_1 is normal to B_1C_1, and

N_2 is an element in B_2C_2 such that A_2N_2 is normal to B_2C_2; and if N_1 is distinct from both B_1 and C_1, then N_2 will be distinct from both B_2 and C_2, and we shall also have

$$A_1N_1 = A_2N_2,$$
$$B_1N_1 = B_2N_2,$$
$$C_1N_1 = C_2N_2.$$

1ST EXTENSION. *The same results hold if, instead of the segments A_1B_1 and A_2B_2 being equal, the general lines A_1B_1 and A_2B_2 are both optical lines.*

2ND EXTENSION. *The same results hold, if in addition to A_1B_1 and A_2B_2 being optical lines, A_1C_1 and A_2C_2 are optical lines instead of the segments A_1C_1 and A_2C_2 being equal.*

It will be observed that, except that we have taken segments instead of pairs, this is the analogue of Theorem 192 and, when none of the sides of the triangles is an optical line, it may be proved in a similar manner using Theorem 181 instead of Theorem 182 in those cases where B_1C_1 is an inertia line and using Theorem 201 instead of Theorem 189. We also must take note of the consequences of Theorem 202 as was done in proving Theorem 205.

As regards the 1st extension it may be proved in a similar manner again taking note of the consequences of Theorem 202 as was done in proving Theorem 205 in the case where A_1B_1 and A_2B_2 are both optical lines.

As regards the 2nd extension it is clear in this case that B_1, A_1 and C_1 are the corners of an optical parallelogram of which B_1C_1 is one diagonal line and N_1 is the centre. Similarly B_2, A_2 and C_2 are the corners of an optical parallelogram of which B_2C_2 is a diagonal line of the same kind as B_1C_1, while N_2 is the centre.

Thus $$B_1N_1 = B_2N_2 = C_1N_1 = C_2N_2,$$
while A_1N_1 is a conjugate to B_1N_1 and A_2N_2 is a conjugate to B_2N_2, so that we must have

$$A_1N_1 = A_2N_2.$$

It is also evident, as in Theorem 192, that we must have

N_2 linearly between B_2 and C_2,

or C_2 linearly between B_2 and N_2,

or B_2 linearly between C_2 and N_2,

according as N_1 is linearly between B_1 and C_1,

or C_1 is linearly between B_1 and N_1,

or B_1 is linearly between C_1 and N_1.

Remarks

If B, C_1 and C_2 be three distinct elements in an inertia plane which do not all lie in one general line, but such that BC_1 and BC_2 are both inertia lines or both separation lines, and if the segments

$$BC_1 = BC_2,$$

while N_1 and N_2 are elements in the general lines BC_1 and BC_2 respectively, such that C_2N_1 is normal to BC_1 and C_1N_2 is normal to BC_2, then we may take the triangle whose corners are B, C_1 and C_2 and apply the results of the last theorem to the one triangle taken in two aspects, as was done in the case of Theorem 192, and so prove that

$$C_2N_1 = C_1N_2,$$
$$BN_1 = BN_2,$$
$$C_1N_1 = C_2N_2.$$

Also the linearly between relations of B, N_2 and C_2 will be similar to those of B, N_1 and C_1 respectively.

Proper hyperbolic angles

While any optical line in a given inertia plane intersects every inertia line and every separation line lying in the inertia plane, this is clearly not the case for every inertia half-line or for every separation half-line in it; since the half-line under consideration may be a part of the complete inertia or separation line which does not contain the element of intersection.

If we have two inertia or two separation half-lines with a common end lying in an inertia plane, such half-lines may be such that they may both be intersected by the same optical lines or such that cannot both be intersected by any optical line.

It should be remembered that, by definition, the end of a half-line is not included in it.

If we revert to the remarks at the end of Theorem 14, employing the notation there used we see that the general half-line OF (where F is any element of b which is *after* E) must be an inertia half-line having F *after* the end element O. Taking any number of positions for F, we get any number of inertia half-lines having the common end O and which are all intersected by the optical line b.

Similarly taking F' as any element of b' which is *before* E' we get an inertia half-line OF' having F' *before* the end element O and, by taking any number of positions for F' we get any number of inertia half-lines

having the common end O and which are all intersected by the optical line b'; which we shall here suppose to be in the same inertia plane as b.

An inertia half-line such as OF and one such as OF' cannot be intersected by the same optical line, for, if F_1 be any element of the half-line OF and F_1' be any element of the half-line OF', then we should have O *after* F_1' and also O *before* F_1 so that if O does not lie in F_1F_1' we know that F_1F_1' could not be an optical line, while if O does lie in F_1F_1' then F_1F_1' must be an inertia line of which the half-lines OF and OF' are parts.

Thus in all cases F_1F_1' must be an inertia line, since F_1 must be *after* F_1'.

Again, since every optical line in the inertia plane intersects every inertia line in it, it follows that every optical line in the inertia plane which intersects one such half-line as OF must intersect all such half-lines; while every optical line in the inertia plane which intersects one such half-line as OF' must intersect all such half-lines.

Again, with the same notation as in the remarks at the end of Theorem 14 we see that any such half-line as OD (where D is any element of b which is *before* E) must be a separation half-line and, taking any number of positions for D we get any number of separation half-lines having the common end O and which are all intersected by the optical line b.

Similarly, any such half-line as OD' (where D' is any element of b' which is *after* E') must be a separation half-line and, taking any number

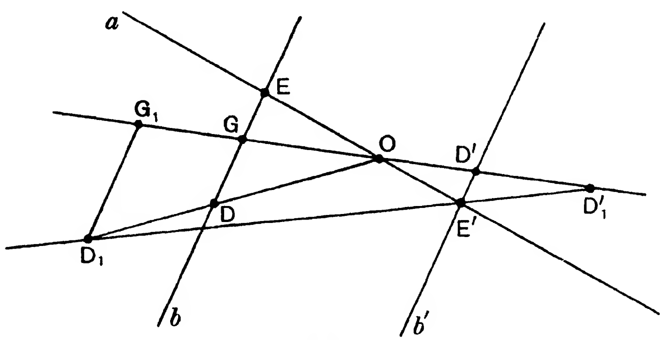


Fig. 54.

of positions for D' we get any number of separation half-lines having the common end O and which are all intersected by the optical line b'.

We shall now prove that if D_1 be any element of the separation half-line OD and D_1' be any element of the separation half-line OD', then

D_1D_1' is a separation line. This is obvious if OD and OD' form parts of the same separation line, so we shall suppose that this is not the case.

Now the complete separation line of which OD' is a part must intersect b in some element, say G.

Also, since O is between the parallel optical lines b and b' and, since D' is *after* E', it follows, by Theorem 69, that E is *after* G.

Since we have supposed that the half-lines OD and OD' are not parts of the same separation line, it follows that G must be distinct from D and may be either *before* or *after* it. The method of proof is similar in the two cases, so that we shall merely consider the case where G is *after* D.

Then, since D_1 and D both lie in the half-line OD which has the same end O as the half-line OG, it follows, by Theorem 67, that, since G is *after* D, an optical line through D_1 co-directional with DG will intersect the half-line OG in an element, say G_1, such that G_1 is *after* D_1.

Now D_1 could not be *after* D_1', for then we should have G_1 *after* D_1'; which is impossible, since G_1D_1' is a separation line.

Again D_1 could not be *before* D_1', for, since O is linearly between D_1' and G_1, it would follow, by Theorem 73, that D_1O must be an inertia line, which it is not.

Thus D_1D_1' must in all cases be a separation line, and no optical line can intersect two such half-lines as OD and OD'.

It follows, as in the case of inertia half-lines, that any optical line in the inertia plane which intersects one such separation half-line as OD must intersect all such separation half-lines, while any optical line in the inertia plane which intersects one such separation half-line as OD' must intersect all such separation half-lines.

Suppose now that we have any two distinct separation half-lines OA and OB having a common end O and lying in an inertia plane, and suppose further that the pair of half-lines are such as may both be intersected by the same optical lines.

Let A and B be any elements of the half-lines and let C be any element which is linearly between A and B.

We shall show that CO must be a separation line.

Let a general line be taken through A parallel to CO. Then this general line must intersect the separation line BO in some element, say H, and since C is linearly between A and B, the element O must be linearly between H and B.

Then the separation half-lines OA and OH having the common end O are such as cannot both be intersected by any optical line and

accordingly AH must be a separation line. Thus, since CO is parallel to AH, it follows that CO must also be a separation line.

Now any general line lying in the inertia plane and normal to CO must be an inertia line. Thus in the particular case where the segment OA = the segment OB and where C is taken to be the mean of A and B, it follows, by Theorem 203, that AB must be normal to CO and therefore AB must be an inertia line.

Thus we get finally that: if OA and OB be two distinct separation lines having a common end O and lying in an inertia plane, and if they are such as can both be intersected by the same optical lines, then if segment OA = segment OB the general line AB must be an inertia line.

If instead of being separation half-lines OA and OB be inertia half-lines such that both A and B are *after* O or else both A and B are *before* O, and if segment OA = segment OB, it follows from the footnote to Theorem 202 that AB must be a separation line.

These are the cases in which the half-lines OA and OB may both be intersected by the same optical lines.

Again, let OA and OA' be two inertia half-lines or two separation half-lines lying in an inertia plane and having a common end O, and such that both half-lines may be intersected by the same optical lines.

Let the elements A and A' be so selected that $OA = OA'$.

Let the two optical lines which pass through A' and lie in the inertia plane intersect the half-line OA in F and G, and let the notation be such that F is linearly between O and G.

Similarly, let the two optical lines which pass through A and lie in the inertia plane intersect the half-line OA' in F' and G', and let the notation be such that F' is linearly between O and G'.

Let N be the mean of F and G while N' is the mean of F' and G'.

Then $A'N$ is normal to AO and AN' is normal to $A'O$.

Also N must lie in the half-line OA, while N' must lie in the half-line OA'.

Further, from the remarks at the end of Theorem 106, it follows that

$$ON = ON'$$

and

$$A'N = AN'.$$

But GN is a conjugate to $A'N$, while $G'N'$ is a conjugate to AN' and so

$$GN = G'N'.$$

Thus, since N must be linearly between O and G, while N' is linearly between O and G', it follows that we must have

$$OG = OG'.$$

Also, since $$OA = OA'$$
and since OA is the hypotenuse of the triangle whose corners are O, N', A, while the side ON' is the same kind of general line as OA, it follows that:

$$OA' < ON'.$$

But $ON' < OG'$ and so A' is linearly between O and G'.

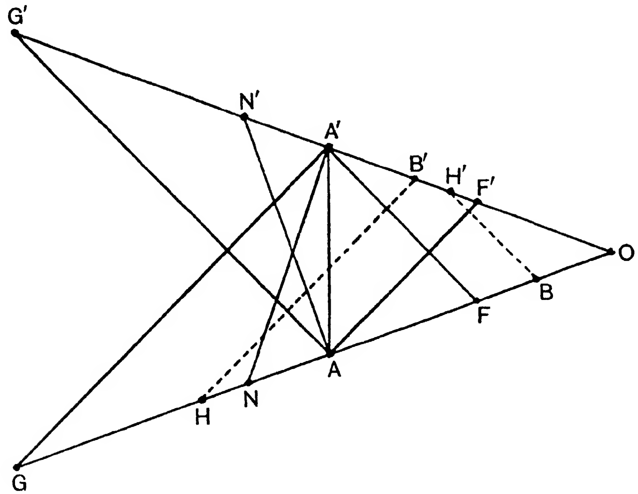


Fig. 55.

Similarly A is linearly between O and G.

Thus, by Theorem 76, the optical lines AG' and $A'G$ intersect and therefore are generators of opposite sets of the inertia plane.

Also, since $OG = OG'$ and $OA = OA'$, it follows that:

$$OG : OA' = OG' : OA.$$

If now BH' be any optical line parallel to AG' and intersecting the half-line OA in B and the half-line OA' in H', while HB' is any optical line parallel to GA' and intersecting the half-line OA in H and the half-line OA' in B'; it follows from the remarks at the end of Theorem 183 that:

$$OH' : OB = OG' : OA$$

and $$OH : OB' = OG : OA'.$$

It follows that if any generator intersects the pair of half-lines and cuts off from them a pair of segments in a particular ratio, then any generator of the same set which intersects the half-lines will cut off segments in the same ratio, while any generator of the opposite set which intersects the half-lines will cut off segments in the reciprocal ratio.

We see from Theorem 202 that these ratios can never be ratios of equality so long as the half-lines OA and OA' are distinct and thus, if expressed numerically, one ratio must be greater than unity, while the other is less than unity.

Let the greater of these reciprocal ratios be z.

Then, since N' is the mean of F' and G', we have

$$ON' = \frac{OG' + OF'}{2},$$

and

$$N'G' = \frac{OG' - OF'}{2}.$$

Thus

$$\frac{ON'}{OA} = \tfrac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{......(1)},$$

while

$$\frac{N'G'}{OA} = \tfrac{1}{2}\left(z - \frac{1}{z}\right) \qquad \text{......(2)}.$$

Suppose now that we have three inertia or three separation half-lines having a common end O and lying in an inertia plane, and such that they may all be intersected by the same optical lines.

Let them be intersected by one such optical line in the elements A_0, A_1 and A_2 respectively and let A_1 be linearly between A_0 and A_2.

Further suppose that this optical line be one for which $\frac{OA_1}{OA_0}$ is greater than unity.

Let the second optical line passing through A_0 and lying in the inertia plane intersect OA_1 in B_1 and OA_2 in B_2.

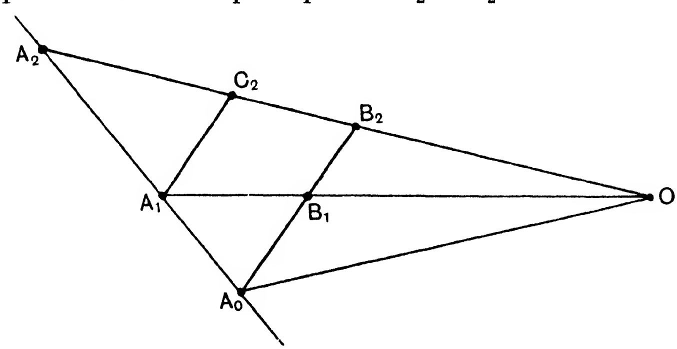


Fig. 56.

Then $OB_1 < OA_1$ so that B_1 is linearly between O and A_1.

But, since A_1 is linearly between A_0 and A_2, it follows, from Theorem 77, that B_2 is linearly between O and A_2 and therefore $OB_2 < OA_2$.

It follows that $\frac{OA_2}{OA_0}$ is greater than unity.

Again let the optical line through A_1 parallel to $A_0 B_2$ intersect OA_2 in C_2.

Then, since A_1 is linearly between A_0 and A_2, it follows that C_2 is linearly between B_2 and A_2 and therefore C_2 is linearly between O and A_2, so that $OC_2 < OA_2$.

Thus $\dfrac{OA_2}{OA_1}$ is greater than unity.

Then $$\frac{OA_2}{OA_0} = \frac{OA_1}{OA_0} \cdot \frac{OA_2}{OA_1};$$

so that taking the logarithms of these ratios we have

$$\log \frac{OA_2}{OA_0} = \log \frac{OA_1}{OA_0} + \log \frac{OA_2}{OA_1}.$$

If, instead of three inertia half-lines or three separation half-lines having the common end O and intersecting the optical line A_0A_1, we have any further number intersecting it in the elements $A_3, A_4, \ldots A_n$ and such that:

A_2 is linearly between A_1 and A_3,
A_3 is linearly between A_2 and A_4,
....
....
A_{n-1} is linearly between A_{n-2} and A_n;

we have $$\log \frac{OA_n}{OA_0} = \log \frac{OA_1}{OA_0} + \log \frac{OA_2}{OA_1} + \ldots + \log \frac{OA_n}{OA_{n-1}}.$$

Reverting now to formulae (1) and (2) and putting $u = \log_e z$ we have $z = e^u$ so that these formulae become

$$\frac{ON'}{OA} = \frac{e^u + e^{-u}}{2} = \cosh u \qquad \text{......(3)},$$

and $$\frac{N'G'}{OA} = \frac{(\text{conjugate } AN')}{OA} = \frac{e^u - e^{-u}}{2} = \sinh u \qquad \text{......(4)},$$

where $$u = \log_e \frac{OG'}{OA} = \log_e \frac{OA}{OF'}.$$

If x_0 and x_1 be two inertia half-lines or two separation half-lines having a common end O and lying in an inertia plane and such that they may both be intersected by the same optical lines, then, together with the element O, they will be said to form a *proper hyperbolic angle-boundary* and the element O will be called its *vertex* while x_0 and x_1 will be called its *sides*.

If A_0 be any element in x_0 and an optical line through A_0 intersects x_1 in A_1, then any general half-line having O as its end and intersecting

A_0A_1 in an element M linearly between A_0 and A_1 must be the same type of half-line as x_0 and x_1.

The set of all such inertia or separation half-lines as OM will be called a *proper hyperbolic angular segment.*

A proper hyperbolic angular segment together with the angle-boundary will be called a *proper hyperbolic angular interval.*

Taking the case of the optical line which makes $OA_1 > OA_0$, then $\log_e \frac{OA_1}{OA_0}$ will be the magnitude of the angular interval in natural measure and will be called a *proper hyperbolic angle.*

Unlike the case of angular intervals in a separation plane there exist proper hyperbolic angular intervals of any magnitude however great.

Other types of angular intervals exist in an inertia plane besides those which we have designated "proper". In these the rays are not all of one kind and it will be found most convenient when we have to deal with them to describe them in terms of proper hyperbolic angular intervals to which they are related. Thus, for example, we may have the supplement or the conjugate of a proper hyperbolic angular interval; or, when one side is an inertia half-line and the other a separation half-line we can construct an auxiliary proper hyperbolic angular interval whose one side is normal to one side of the given one.

Various results may be deduced by means of formulae (3) and (4) analogous to theorems in ordinary geometry. Thus if A, B and C be the corners of a triangle in an inertia plane and if we denote the sides by a, b and c as in ordinary geometry; then if the sides b and c form a proper hyperbolic angle-boundary with one another which we denote by A, and we form the expression

$$b^2 + c^2 - 2bc \cosh A,$$

then, if this expression be positive, the side a will be a separation segment if b and c are separation segments and will be an inertia segment if b and c are inertia segments and in either case we shall have

$$b^2 + c^2 - 2bc \cosh A = a^2.$$

If the expression be zero, then a will be a segment of an optical line.

If the expression be negative, then the side a will be an inertia segment if b and c are separation segments and will be a separation segment if b and c are inertia segments and in either case we shall have

$$2bc \cosh A - b^2 - c^2 = (\text{conjugate } a)^2.$$

These results may easily be deduced and we shall not trouble to prove them.

It should be noted however that when the side a is an optical segment, nothing is said as to its length and we may even have two triangles with two sides and the included proper hyperbolic angle of the one respectively equal to the two sides and the included proper hyperbolic angle of the other while the third side of the one forms a part of the third side of the other; but this can only happen if these third sides are optical segments.

On the other hand, no comparison whatever can be made in the lengths of the third sides if they be optical segments but not co-directional.

Introduction of coordinates

If we take any element O of the set as origin, we have already seen that we may obtain systems of four general lines through O, say OX, OY, OZ, OT, which are mutually normal to one another.

Three of these, say OX, OY, OZ, will be separation lines, while the fourth, OT, will be an inertia line.

The three separation lines OX, OY, OZ will determine a separation threefold, say W, and OT will be normal to it.

If we select any arbitrary separation segment as a unit of length and associate the number zero with the element O, we may associate every other element of OX, OY, OZ with a real number, positive or negative, corresponding to the length of the segment of which that element is one end and the origin O is the other.

In this way we set up a coordinate system in W which will be quite similar to that with which we are familiar.

Since all the theorems of ordinary Euclidean geometry hold for a separation threefold, the length of a segment in W will be given by the ordinary Cartesian formula.

Again, nor confining our attention merely to the elements of W, let A be any element of the whole set.

Then A must either lie in OT, or else there is an inertia line through A parallel to OT, and, as has already been proved, this inertia line will intersect W in some element, say N.

Further, AN must be normal to W.

Now if A does not lie in W there will be a separation threefold, say W', passing through A and parallel to W, and the inertia line OT must intersect W' in some element, say M.

Further, since W' is parallel to W, both OT and AN must be normal to W'.

Thus, if OM and NA are distinct, MA and ON must both be separation lines normal to OM, and so, since OM and NA lie in an inertia plane, we must have MA parallel to ON.

Now we may select a unit inertia segment, just as we selected a unit separation segment, and with each element of OT distinct from O we may associate a real number positive or negative corresponding to the length of the segment of which that element is one end and the origin O is the other.

We shall suppose this correspondence to be set up in such a way that a positive real number corresponds to any element which is after O *and a negative real number to any element which is* before O.

As regards the relationship between the unit separation segment and the unit inertia segment, the simplest convention to make is to take the unit inertia segment such that its conjugate is equal to the unit separation segment.

More generally, we may take the unit inertia segment such that:

(conjugate of unit inertia segment) $= v$ (unit separation segment),

where v is a constant afterwards to be identified with what we call the "*velocity of light*".

Now the element N lies in W and is determined by three coordinates, say x_1, y_1, z_1, taken parallel to OX, OY, OZ respectively in the usual manner.

Further segment $NA =$ segment OM,

and so if t_1 be the length of OM in terms of the unit inertia segment, then the element A will be determined by the four coordinates x_1, y_1, z_1, t_1.

Let the length of the segment ON be denoted by w_1.

Then as in ordinary coordinate geometry

$$w_1^2 = x_1^2 + y_1^2 + z_1^2.$$

Thus if OA should be an optical line, we must have

$$w_1^2 = v^2 t_1^2,$$

or

$$x_1^2 + y_1^2 + z_1^2 - v^2 t_1^2 = 0 \qquad \ldots\ldots(1).$$

Again, if OA should be a separation segment and if r_1 be its length, it follows from the analogue of the theorem of Pythagoras for this case that:

$$w_1^2 - v^2 t_1^2 = r_1^2,$$

or

$$x_1^2 + y_1^2 + z_1^2 - v^2 t_1^2 = r_1^2 \qquad \ldots\ldots(2).$$

Finally, if OA should be an inertia segment and $\bar{r}_1$ its length, it

follows from the corresponding analogue of the Pythagoras theorem that:

$$w_1^2 - v^2 t_1^2 = -v^2 \bar{r}_1^2,$$

or $$x_1^2 + y_1^2 + z_1^2 - v^2 t_1^2 = -v^2 \bar{r}_1^2 \qquad \ldots\ldots(3).$$

Thus from (1), (2) *and* (3) *it follows that the expression*

$$x_1^2 + y_1^2 + z_1^2 - v^2 t_1^2$$

is positive, zero, or negative according as OA is a separation line, an optical line, or an inertia line.

If A be *after* O, it is clear from the convention which we have made that t_1 must be positive, and so the conditions that A should be *after* O are:

$$\left.\begin{array}{ll} (1) & x_1^2 + y_1^2 + z_1^2 - v^2 t_1^2 \text{ is zero or negative} \\ (2) & t_1 \text{ is positive} \end{array}\right\}.$$

The conditions that A should be *before* O are similarly:

$$\left.\begin{array}{ll} (1) & x_1^2 + y_1^2 + z_1^2 - v^2 t_1^2 \text{ is zero or negative} \\ (2) & t_1 \text{ is negative} \end{array}\right\}.$$

The conditions that A should be neither *before* nor *after* O are either that:

$$A \text{ is identical with } O,$$

$$\left.\begin{array}{ll} \text{in which case} & x_1 = y_1 = z_1 = t_1 = 0 \\ \text{or else} & x_1^2 + y_1^2 + z_1^2 - v^2 t_1^2 \text{ is positive} \end{array}\right\}.$$

More generally, it is clear that: if (x_0, y_0, z_0, t_0) and $(x_1, y_1, z_1, t)_1$ be the coordinates of two elements which we call A_0 and A_1 respectively, then if A_0 and A_1 lie in an optical line we must have

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 - v^2 (t_1 - t_0)^2 = 0 \quad \ldots\ldots(4).$$

If $A_0 A_1$ be a separation segment and r_1 be its length we must have

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 - v^2 (t_1 - t_0)^2 = r_1^2 \quad \ldots\ldots(5).$$

While if $A_0 A_1$ be an inertia segment and $\bar{r}_1$ be its length we must have

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 - v^2 (t_1 - t_0)^2 = -v^2 \bar{r}_1^2 \quad \ldots(6).$$

Thus the expression

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 - v^2 (t_1 - t_0)^2$$

is positive, zero, or negative according as $A_0 A_1$ *is a separation line, an optical line, or an inertia line.*

Accordingly if A_0 and A_1 be any elements of the set, the conditions that A_1 should be *after* A_0 are:

$$\left.\begin{array}{lll} & (1) & (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 - v^2 (t_1 - t_0) \\ & & \qquad\qquad\qquad \text{is zero or negative} \\ \text{and} & (2) & t_1 - t_0 \text{ is positive} \end{array}\right\}.$$

The conditions that A_1 should be *before* A_0 are:

$$\left.\begin{array}{ll}(1) & (x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2-v^2(t_1-t_0)^2 \\ & \qquad\qquad\qquad\qquad \text{is zero or negative} \\ \text{and}\quad (2) & \qquad t_1-t_0 \text{ is negative}\end{array}\right\}.$$

The conditions that A_1 should be neither *before* nor *after* A_0 are (if we include the case where A_0 and A_1 are identical):

$$\left.\begin{array}{l} x_1-x_0=y_1-y_0=z_1-z_0=t_1-t_0=0 \\ \text{or else} \\ (x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2-v^2(t_1-t_0)^2 \text{ is positive}\end{array}\right\}.$$

Now the condition that two distinct elements lie in an optical line gives us also the condition that the one should lie in the α sub-set of the other.

Thus if (x_0, y_0, z_0, t_0) be the coordinates of an element A_0, the equation of the combined α and β sub-sets of A_0 is

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2-v^2(t-t_0)^2=0 \qquad \ldots\ldots(7).$$

The α sub-set will then consist of all elements (x, y, z, t) for which this equation is satisfied and for which $t-t_0$ is zero or positive; while the β sub-set of A_0 will consist of all elements for which the equation is satisfied and for which $t-t_0$ is zero or negative.

Definition. The set of all elements whose coordinates satisfy equation (7) will be called the *standard cone* with respect to the element whose coordinates are (x_0, y_0, z_0, t_0).

Taking v equal to unity, for the sake of simplicity, it is evident that the equation

$$x^2+y^2+z^2-t^2=c^2$$

represents the set of elements such as A, where OA is a separation segment whose length is c.

Similarly, the equation

$$x^2+y^2+z^2-t^2=-c^2$$

represents the set of elements such as A, where OA is an inertia segment whose length is c.

If we put $y=0$ and $z=0$ in the first of these we obtain

$$x^2-t^2=c^2,$$

which gives us the relation between x and t for the portion of the corresponding set which lies in the inertia plane containing the axes of x and t.

This then represents the analogue of a circle in the inertia plane.

Similarly for the case of inertia segments, putting $y=0$ and $z=0$ we get

$$x^2-t^2=-c^2.$$

The two equations:

$$x^2-t^2=c^2 \quad \text{and} \quad x^2-t^2=-c^2$$

are of the same forms as the equations of a hyperbola and its conjugate in ordinary plane geometry.

The equation
$$x^2-t^2=0$$
along with $y=0$ and $z=0$ represents the two optical lines through the origin in the same inertia plane, and these correspond to the common asymptotes of the hyperbolas.

Normality of general lines

Let A, B and C be three distinct elements whose coordinates are (x_0, y_0, z_0, t_0), (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) respectively and such that AB is normal to AC.

For the sake of simplicity we shall take v equal to unity.

For brevity let us write

$$(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2-(t_2-t_1)^2=H,$$
$$(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2-(t_1-t_0)^2=S_1,$$
$$(x_2-x_0)^2+(y_2-y_0)^2+(z_2-z_0)^2-(t_2-t_0)^2=S_2.$$

Considering all the six cases of analogue to the theorem of Pythagoras (including the limiting case mentioned in the footnote on p. 369), we see that they are all included in the formula:

$$H=S_1+S_2;$$

or expanding, rearranging and omitting a factor 2, in the formula:

$$(x_2-x_0)(x_1-x_0)+(y_2-y_0)(y_1-y_0)$$
$$+(z_2-z_0)(z_1-z_0)-(t_2-t_0)(t_1-t_0)=0. \quad (1)$$

We have to show that this is not merely a necessary, but also a sufficient condition of the normality of AB to AC.

Now let us consider the various possibilities which are conceivable with regard to the types of general line which AB, AC and BC might be.

It is obvious that the two sides of the equation

$$H=S_1+S_2$$

must either be both positive, both negative, or both zero.

For H positive, it is clear that we could only conceivably have S_1 and S_2 either (1) both positive, or (2) one positive and other negative, or (3) one positive and other zero.

For H negative, we could only conceivably have S_1 and S_2 either (4) both negative, or (5) one negative and other positive, or (6) one negative and other zero.

For H zero, we could only conceivably have S_1 and S_2 either (7) both zero, or (8) one positive and other negative.

However, cases (4) and (6) can be shown to be impossible from other considerations.

We showed geometrically that this was so, but it is desirable to show the reasons analytically.

In order to do so we shall first investigate a certain lemma.

Suppose that we have two series of four corresponding quantities, say $Q_1, \quad Q_2, \quad Q_3, \quad Q_4,$
and $R_1, \quad R_2, \quad R_3, \quad R_4.$

Then the following identity may easily be verified:

$$\begin{aligned}&(R_1 \pm R_4)^2\{Q_1^2+Q_2^2+Q_3^2-Q_4^2\}+(Q_1 \pm Q_4)^2\{R_1^2+R_2^2+R_3^2-R_4^2\}\\&-2(Q_1 \pm Q_4)(R_1 \pm R_4)\{Q_1R_1+Q_2R_2+Q_3R_3-Q_4R_4\}\\&\equiv\{Q_2R_1-Q_1R_2 \pm (Q_2R_4-Q_4R_2)\}^2+\{Q_3R_1-Q_1R_3 \pm (Q_3R_4-Q_4R_3)\}^2;\end{aligned}$$

where, in the ambiguities, either the positive sign is to be used throughout, or else the negative sign throughout.

In this identity we may obviously interchange R_1 with either R_2 or R_3, while at the same time we interchange Q_1 with Q_2 or Q_3 respectively.

We shall now prove that if

$$Q_1R_1+Q_2R_2+Q_3R_3-Q_4R_4=0,$$

while $Q_1^2+Q_2^2+Q_3^2-Q_4^2$ is negative and equal to $-k^2$, then

$$R_1^2+R_2^2+R_3^2-R_4^2$$

must be positive unless

$$R_1=R_2=R_3=R_4=0;$$

when it is zero.

Let $$R_1^2+R_2^2+R_3^2-R_4^2=\theta,$$

and our identity gives us

$$\begin{aligned}(Q_1 \pm Q_4)^2\theta=(R_1 \pm R_4)^2k^2&+\{Q_2R_1-Q_1R_2 \pm (Q_2R_4-Q_4R_2)\}^2\\&+\{Q_3R_1-Q_1R_3 \pm (Q_3R_4-Q_4R_3)\}^2.\end{aligned}$$

If $R_1 \pm R_4$ be not zero, the right-hand side of this equation must be positive and therefore neither $(Q_1 \pm Q_4)^2$ nor θ can be zero, and, since $(Q_1 \pm Q_4)^2$ must be positive, therefore θ must be positive.

Thus θ must be positive if R_1^2 is not equal to R_4^2.

By the use of similar identities we may prove that θ must be positive

if R_2^2 is not equal to R_4^2 and also if R_3^2 is not equal to R_4^2. Thus θ must be positive unless

$$R_1^2 = R_2^2 = R_3^2 = R_4^2.$$

But in this case we should have

$$\theta = R_1^2 + R_2^2 + R_3^2 - R_4^2 = 2R_4^2,$$

which is positive unless $R_4 = 0$; when it is zero.

Thus $R_1^2 + R_2^2 + R_3^2 - R_4^2$ must always be positive unless

$$R_1 = R_2 = R_3 = R_4 = 0.$$

If now we substitute

$$(x_1 - x_0),\quad (y_1 - y_0),\quad (z_1 - z_0),\quad (t_1 - t_0)$$

for Q_1, Q_2, Q_3, Q_4 respectively, and

$$(x_2 - x_0),\quad (y_2 - y_0),\quad (z_2 - z_0),\quad (t_2 - t_0)$$

for R_1, R_2, R_3, R_4 respectively, and suppose that equation (1) (which is equivalent to $H = S_1 + S_2$) holds and that

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 - (t_1 - t_0)^2$$

is negative; then

$$(x_2 - x_0)^2 + (y_2 - y_0)^2 + (z_2 - z_0)^2 - (t_2 - t_0)^2$$

must be positive or zero and, in the latter case, we must have

$$(x_2 - x_0) = (y_2 - y_0) = (z_2 - z_0) = (t_2 - t_0) = 0,$$

or C coincident with A, contrary to the hypothesis that C and A are distinct.

Thus cases (4) and (6) are both excluded as possibilities, and we are left only with cases (1), (2), (3), (5), (7), (8), which are precisely the six cases considered in the remarks at the end of Theorem 205.

Thus, provided that equation (1) holds, the general line AB must be normal to the general line AC.

We may also make use of our identity in order to obtain the equations of an optical line from the definition give on p. 30 and incidentally to give an analytical demonstration that, in case (7) above, the three elements, A, B and C, lie in one optical line.

Thus putting

$$\begin{aligned} Q_1R_1 + Q_2R_2 + Q_3R_3 - Q_4R_4 &= 0, \\ Q_1^2 + Q_2^2 + Q_3^2 - Q_4^2 &= 0, \\ R_1^2 + R_2^2 + R_3^2 - R_4^2 &= 0, \end{aligned}$$

in our identity, we get

$$0 = \{Q_2R_1 - Q_1R_2 \pm (Q_2R_4 - Q_4R_2)\}^2 + \{Q_3R_1 - Q_1R_3 \pm (Q_3R_4 - Q_4R_3)\}^2.$$

Since the right-hand side is a sum of two squares equated to zero, they must be each separately zero, and so

$$Q_2 R_1 - Q_1 R_2 \pm (Q_2 R_4 - Q_4 R_2) = 0,$$
$$Q_3 R_1 - Q_1 R_3 \pm (Q_3 R_4 - Q_4 R_3) = 0,$$

and, since either the $+$ or $-$ sign may be taken in the ambiguities, we see that:

$$Q_2 R_1 - Q_1 R_2 = 0,\ Q_2 R_4 - Q_4 R_2 = 0,\ Q_3 R_1 - Q_1 R_3 = 0,\ Q_3 R_4 - Q_4 R_3 = 0$$

or
$$Q_1 : Q_2 : Q_3 : Q_4 = R_1 : R_2 : R_3 : R_4.$$

Now let A and B be two distinct elements such that the one lies in the α sub-set of the other and let (x_0, y_0, z_0, t_0) and (x_1, y_1, z_1, t_1) be the coordinates of A and B respectively.

Then $$(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 - (t_1 - t_0)^2 = 0.$$

If (x, y, z, t) be any element which lies in the standard cones with respect to both A and B, we know from definition that such element lies in the optical line containing A and B.

But in this case we must have

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - (t - t_0)^2 = 0,$$

and
$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 - (t - t_1)^2 = 0.$$

It follows that we must have

$$(x - x_0)(x_1 - x_0) + (y - y_0)(y_1 - y_0) + (z - z_0)(z_1 - z_0) - (t - t_0)(t_1 - t_0) = 0.$$

Thus substituting

$$(x_1 - x_0),\quad (y_1 - y_0),\quad (z_1 - z_0),\quad (t_1 - t_0)$$

for Q_1, Q_2, Q_3, Q_4 respectively and

$$(x - x_0),\quad (y - y_0),\quad (z - z_0),\quad (t - t_0)$$

for R_1, R_2, R_3, R_4 respectively, the result above obtained enables us to write

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0} = \frac{t - t_0}{t_1 - t_0};$$

which are the equations of the optical line containing A and B.

If C be any element distinct from both A and B and such that both AC and BC are optical segments and if (x_2, y_2, z_2, t_2) be the coordinates of C, it is evident that (x_2, y_2, z_2, t_2) satisfies the above conditions and accordingly, C must lie in the optical line through A and B. This is equivalent to case (7).

The above identity may be generalised to n dimensions and will be found very useful in the further analytical development of this subject. See paper by the author "On the Connexion of a Certain Identity with

the Extension of Conical Order to n Dimensions", *Camb. Phil. Soc.* vol. XXIV, pp. 357–74, 1928.

EQUATIONS OF GENERAL LINES, PLANES AND THREEFOLDS

Making use of the notation employed in the preceding section let us take the elements A and B as fixed while C is variable and substituting the running coordinates (x, y, z, t) for (x_2, y_2, z_2, t_2) in equation (1) we get

$$(x-x_0)(x_1-x_0)+(y-y_0)(y_1-y_0)+(z-z_0)(z_1-z_0)-(t-t_0)(t_1-t_0)=0, \qquad \ldots\ldots(2)$$

as the equation of the general threefold passing through A and normal to AB.

This will be an inertia, an optical, or a separation threefold according as AB is a separation, an optical or an inertia line. That is to say, equation (2) will represent an inertia, an optical, or a separation threefold according as the expression

$$(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2-(t_1-t_0)^2$$

is positive, zero, or negative.

If l, m, n, p be any four quantities such that:

$$\frac{x_1-x_0}{l}=\frac{y_1-y_0}{m}=\frac{z_1-z_0}{n}=\frac{t_1-t_0}{p},$$

then $$l(x-x_0)+m(y-y_0)+n(z-z_0)-p(t-t_0)=0 \qquad \ldots\ldots(3)$$

will obviously represent the same set of elements as equation (2) and is therefore the equation of a threefold passing through the element whose coordinates are (x_0, y_0, z_0, t_0), and its type will be, inertia, optical, or separation, according as the expression

$$l^2+m^2+n^2-p^2$$

is positive, zero or negative.

If (x', y', z', t') be any element such that:

$$\frac{x'-x_0}{x_1-x_0}=\frac{y'-y_0}{y_1-y_0}=\frac{z'-z_0}{z_1-z_0}=\frac{t'-t_0}{t_1-t_0},$$

then we could substitute

$$(x'-x_0),\ (y'-y_0),\ (z'-z_0),\ (t'-t_0)$$

for $$(x_1-x_0),\ (y_1-y_0),\ (z_1-z_0),\ (t_1-t_0)$$

respectively in equation (2) and the resultant equation would still represent the same threefold while the general line through (x_0, y_0, z_0, t_0) and (x', y', z', t') would still be the normal to the threefold through the element A which, as we know, is unique.

Thus (x', y', z', t') would always lie in the general line AB.

Thus, removing the accents we get

$$\frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0}=\frac{z-z_0}{z_1-z_0}=\frac{t-t_0}{t_1-t_0} \quad \ldots\ldots(4)$$

as the equations of the general line passing through the two elements (x_0, y_0, z_0, t_0) and (x_1, y_1, z_1, t_1); and these will represent a separation, an optical, or an inertia line according as the expression

$$(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2-(t_1-t_0)^2$$

is positive, zero, or negative.

As before, we may substitute l, m, n, p for

$$(x_1-x_0), (y_1-y_0), (z_1-z_0), (t_1-t_0)$$

respectively and the equations

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}=\frac{t-t_0}{p} \quad \ldots\ldots(5)$$

will represent the same general line, which will be separation, optical, or inertia according as the expression

$$l^2+m^2+n^2-p^2$$

is positive, zero, or negative.

Again, if (x_2, y_2, z_2, t_2) be any element of the general threefold (2) distinct from the element (x_0, y_0, z_0, t_0), then the general line joining these elements will be normal to the general line AB and

$$(x_2-x_0)(x_1-x_0)+(y_2-y_0)(y_1-y_0)+(z_2-z_0)(z_1-z_0)-(t_2-t_0)(t_1-t_0) = 0.$$

This will be the condition that the lines are normal to one another.

If the general lines be expressed in the forms:

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}=\frac{t-t_0}{p}$$

and

$$\frac{x-x_0}{l'}=\frac{y-y_0}{m'}=\frac{z-z_0}{n'}=\frac{t-t_0}{p'}$$

the condition of normality may be expressed in the form

$$ll'+mm'+nn'-pp'=0 \quad \ldots\ldots(6).$$

It can readily be seen that this is still the condition of normality if the lines do not both pass through the same element.

Since any two general threefolds which are not parallel have a general plane in common, it follows that any two equations of the forms:

$$l_1x+m_1y+n_1z-p_1t=c_1 \quad \ldots\ldots(7)$$

and

$$l_2x+m_2y+n_2z-p_2t=c_2 \quad \ldots\ldots(8)$$

where $$l_1 : m_1 : n_1 : p_1 \neq l_2 : m_2 : n_2 : p_2,$$
will represent some form of general plane P.

In order to determine to which type this belongs consider the two general lines:

$$\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1} = \frac{t}{p_1} = r \qquad \ldots\ldots(9),$$

$$\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2} = \frac{t}{p_2} = r \qquad \ldots\ldots(10).$$

These general lines both pass through the origin and therefore lie in some general plane Q.

Further, since the general line (9) must be normal to the general threefold (7), while (10) is normal to (8), it follows that (9) and (10) are both normal to P and consequently Q must be completely normal to P.

We shall determine the type of Q and thence deduce the type of P.

Let (x_1, y_1, z_1, t_1) be any element in (9) distinct from the origin.

Then the equation of a standard cone having this element as vertex will be

$$(x - l_1 r_1)^2 + (y - m_1 r_1)^2 + (z - n_1 r_1)^2 - (t - p_1 r_1)^2 = 0.$$

If this cone intersects the general line (10) in an element (x_2, y_2, z_2, t_2), we shall have

$$(l_2 r_2 - l_1 r_1)^2 + (m_2 r_2 - m_1 r_1)^2 + (n_2 r_2 - n_1 r_1)^2 - (p_2 r_2 - p_1 r_1)^2 = 0$$

or

$$(l_2^2 + m_2^2 + n_2^2 - p_2^2) r_2^2 - 2(l_2 l_1 + m_2 m_1 + n_2 n_1 - p_2 p_1) r_1 r_2 + (l_1^2 + m_1^2 + n_1^2 - p_1^2) r_1^2 = 0.$$

Let us first suppose that neither of the expressions:

$$(l_2^2 + m_2^2 + n_2^2 - p_2^2)$$
$$(l_1^2 + m_1^2 + n_1^2 - p_1^2)$$

is zero.

Regarded as an equation for r_2 in terms of r_1 and the direction ratios, the condition that the roots should be real and distinct is that

$$(l_2 l_1 + m_2 m_1 + n_2 n_1 - p_2 p_1)^2 - (l_2^2 + m_2^2 + n_2^2 - p_2^2)(l_1^2 + m_1^2 + n_1^2 - p_1^2) > 0.$$

If this be the case the general plane Q will be such that it contains two optical lines passing through an element of it.

It follows that Q will be an inertia plane.

If the above expression be zero there will be only one optical line in Q which passes through the given element and accordingly, in this case, Q will be an optical plane.

If the above expression be negative there will be no optical line in Q which passes through the given element and accordingly, in this case, Q will be a separation plane.

Let us next consider the case where one of the general lines (9) and (10) is an optical line. It will be sufficient to suppose that

$$l_2^2 + m_2^2 + n_2^2 - p_2^2 = 0$$

while

$$l_1^2 + m_1^2 + n_1^2 - p_1^2 \neq 0.$$

Then provided that the expression

$$l_2 l_1 + m_2 m_1 + n_2 n_1 - p_2 p_1$$

be not zero, r_2 may be determined and Q contains a second optical line which intersects the optical line (l_2, m_2, n_2, p_2) and accordingly Q will be an inertia plane.

It is obvious however that in this case also

$$(l_2 l_1 + m_2 m_1 + n_2 n_1 - p_2 p_1)^2 \\ - (l_2^2 + m_2^2 + n_2^2 - p_2^2)(l_1^2 + m_1^2 + n_1^2 - p_1^2) > 0.$$

In case $l_2 l_1 + m_2 m_1 + n_2 n_1 - p_2 p_1$ be zero, the above expression is zero and no value of r_2 can be determined. Thus in this case there is no optical line in Q which will intersect the optical line (l_2, m_2, n_2, p_2); so that Q must be an optical plane.

Finally if

$$l_2^2 + m_2^2 + n_2^2 - p_2^2 = 0$$

and

$$l_1^2 + m_1^2 + n_1^2 - p_1^2 = 0,$$

there are evidently two intersecting optical lines in Q, which must therefore be an inertia plane.

Since these optical lines intersect it is not possible to have

$$l_2 l_1 + m_2 m_1 + n_2 n_1 - p_2 p_1$$

zero: for this would be the condition of their normality which would imply coincidence.

Thus in this case also we should have

$$(l_2 l_1 + m_2 m_1 + n_2 n_1 - p_2 p_1)^2 \\ - (l_2^2 + m_2^2 + n_2^2 - p_2^2)(l_1^2 + m_1^2 + n_1^2 - p_1^2) > 0.$$

Accordingly in all cases this expression will be positive, zero or negative, according as Q is an inertia plane, an optical plane or a separation plane.

But since P is completely normal to Q, it follows that when:

(i) Q is an inertia plane, P is a separation plane.
(ii) Q is an optical plane, P is an optical plane.
(iii) Q is a separation plane, P is an inertia plane.

Thus P is a separation plane, an optical plane or an inertia plane according as the expression

$$(l_2 l_1 + m_2 m_1 + n_2 n_1 - p_2 p_1)^2 - (l_2^2 + m_2^2 + n_2^2 - p_2^2)(l_1^2 + m_1^2 + n_1^2 - p_1^2)$$

is positive, zero, or negative.

Having thus obtained the equations of general lines, planes and threefolds and the conditions that they should be of any one of the three different types and having also obtained the condition that general lines should be normal to one another; the analytical development of the subject may be carried forward in the usual manner.

Symmetrical coordinates

The systems of coordinates which we have considered are those in which we have four coordinate axes which are normal to one another, and such systems are those which are most generally useful; but they give an expression for the square of the distance between two elements in the form of a sum of squares, in which one square is of different sign from the remaining ones.

This want of symmetry takes away from the analytical attractiveness of the subject, although the fact that it can be built up entirely from *before* and *after* relations gives it a special importance.

The writer has shown that it is possible to introduce symmetrical systems of coordinates for Conical Order in four or any larger number of dimensions; which, however, are not orthogonal.*

Thus if we put:

$$(X_1 + X_2 - X_3 - X_4)/\sqrt{6} = x,$$
$$(X_1 - X_2 + X_3 - X_4)/\sqrt{6} = y,$$
$$(X_1 - X_2 - X_3 + X_4)/\sqrt{6} = z,$$
$$\sqrt{3}\,(X_1 + X_2 + X_3 + X_4)/\sqrt{6} = t,$$

we can easily verify that:

$$t^2 - x^2 - y^2 - z^2 = \tfrac{4}{3}\{X_1 X_2 + X_2 X_3 + X_3 X_1 + X_1 X_4 + X_2 X_4 + X_3 X_4\};$$

which is symmetrical in the four coordinates: X_1, X_2, X_3, X_4.

* In order to construct a conical order of five dimensions it is only necessary to omit Post. XX, and substitute for it a postulate of the form:

If W be any optical threefold there is at least one element which is neither before *nor* after *any element of W.*

The subject can then very easily be developed, since the main difficulties have already been overcome in treating of four dimensions. We may limit the geometry to five dimensions, if so desired, by means of a postulate analogous to Post. XX; or extend it to six dimensions or any larger number in an analogous manner.

To go into any further details on this subject would be outside the scope of the present work, which is concerned with the development in four dimensions.

The transformation has been made of such a form as to introduce a coefficient $\frac{4}{3}$ for reasons connected with the interpretation of the system.

We do not purpose going into this in the present work, but refer the reader to a paper by the author: "On a Symmetrical Analysis of Conical Order and its Relation to Time-Space Theory in the *Proc. Roy. Soc.* A (1930), vol. cxxix, pp. 549–79.

Interpretation of results

It is evident that any element whose coordinates are $(a, b, c, 0)$ must lie in the separation threefold W and accordingly the three equations

$$x=a, \quad y=b, \quad z=c$$

must represent an inertia line normal to W and therefore co-directional with the axis of t.

Again, any equation of the first degree in x, y, z, together with the equation $t=0$, will represent a separation plane in W, while any two independent but consistent equations of the first degree in x, y, z, together with the equation $t=0$, will represent a separation line in W.

Thus any equation of the first degree in x, y, z (leaving out the equation $t=0$) will represent an inertia threefold containing inertia lines parallel to the axis of t; while any two independent but consistent equations of the first degree in x, y, z will represent an inertia plane containing inertia lines parallel to the axis of t.

Thus corresponding to any theorem concerning the elements of W there will be a theorem concerning inertia lines normal to W and passing through these elements.

Conversely, if we consider the system consisting of any selected inertia line together with all others parallel to it, then any two such inertia lines will determine an inertia plane, while any three which do not lie in one inertia plane will determine an inertia threefold.

Since these inertia lines must all intersect any separation threefold to which they are normal, it follows that they have a geometry similar to that of the separation threefold and therefore of the ordinary Euclidean type.

If then we call any element of the entire set an "instant"; any inertia line of the selected system a "point"; any inertia plane of the selected system a "straight line"; and any inertia threefold of the selected system a "plane"; we can speak of succeeding instants at any given point, and have thus obtained a representation of the space and time of our experience in so far as their geometrical relations are concerned.

The distance between two parallel inertia lines of the system will naturally be taken as the length of the segment intercepted by them in a separation line which intersects them both normally.

This, then, will be the meaning to be attached to the *distance between two points*.

Time intervals in the usual sense will be measured by the lengths of segments of the corresponding inertia lines: that is to say, by differences of the t coordinates.

Since we have defined the equality of separation and inertia segments in terms of the relations of *after* and *before* and have assigned an interpretation of these, it follows that the equality of length and time intervals in the ordinary sense is rendered precise.

It is to be observed that the particular system of parallel inertia lines which we may select is quite arbitrary although the set of elements or instants contained in the entire system is in all cases identical.

The distinction between different systems is that while two parallel inertia lines represent the time paths of unaccelerated particles which are at rest relative to one another; two non-parallel inertia lines represent the time paths of unaccelerated particles which are in motion with uniform velocity with respect to one another.

Thus we are able to give a definition of absence of acceleration, but, since all inertia lines are on a par with one another, we can attach no meaning to a particle or system being at "absolute rest".

The definition of absence of acceleration based upon the relations of *after* and *before* and as regards a finite interval of time, may be thus expressed:

Definition. If A and B be two distinct elements of any inertia line (B being *after* A), then a particle will be said to be *unaccelerated from the instant* A *to the instant* B provided it lies in the inertia line AB throughout that interval.

The physical signification of an optical line is: that a flash of light or other instantaneous electromagnetic disturbance in going directly from one particle to another would follow this time path.

As regards a separation line; since no element of it is either *before* or *after* another, then if our view be correct, no single particle could occupy more than one element, and so particles which occupy distinct elements of any separation line must be separate particles.

The above considerations indicate the reasons for adopting the names we have assigned to the three types of general line.

The names inertia, optical and separation, as applied to general planes and general threefolds, have been given on account of certain analogies with the corresponding types of general lines.

In the first edition of this work the names: "acceleration plane" and "rotation threefold" were used instead of inertia plane and inertia threefold respectively; but the present nomenclature is more systematic and permits of systematic extension to Conical Order in n dimensions.

Results involving only three coordinates x, y and t may be visualised by means of the three-dimensional conical order described in the introduction, but a certain amount of distortion appears in a model of this kind, since equal lengths in the model do not in general represent equal lengths as we have defined them.

The optical significations of Posts. I to XVIII are however made clear by such models, and it is easily seen that the assertions made in these postulates, when interpreted in the manner described, are in accordance with the ordinarily accepted ideas.

Post. XXI also finds an interpretation in such a model, but its significance is concerned rather with the logic of continuity than with any observable physical phenomenon.

Since it is possible to define equality of lengths in terms of *after* and *before* it seems superfluous to introduce any other conception of length, since the effect of this would merely be to destroy the symmetry which otherwise exists.

It is again to be emphasised that the application of the theory of conical order does not in itself require that the α and β sub-sets should be determined by optical phenomena, but merely that there should exist some influence having the properties which we have ascribed to light.

Accordingly if it should be found hereafter that some other influence than light possessed these properties we should merely require to substitute this influence for light and interpret our results in terms of it.

Conclusion

Our task now approaches completion.

We have shown how from some twenty-one postulates involving the ideas of *after* and *before* it is possible to set up a system of geometry in which any element may be represented by four coordinates x, y, z, t.

Three of these, x, y, z, correspond to what we ordinarily call space

coordinates, while the fourth corresponds to time as generally understood.

Since however an element in this geometry corresponds to an instant, and bears the relations of *after* and *before* to certain other instants, it appears that the theory of space is really a part of the theory of time.

Of the postulates used: nineteen, namely I to XVIII and Post. XXI, may easily be seen to have an interpretation in three-dimensional geometry by making use of cones as described in the introduction.

It follows that if ordinary geometry be consistent with itself, these nineteen postulates must be consistent with one another.

Of the remaining two postulates, Post. XIX has the effect of introducing one more dimension, while Post. XX limits the number of dimensions to four.

Since by means of these we have been enabled to set up a coordinate system in the four variables x, y, z, t, the question of the consistency of the whole twenty-one postulates is reduced to analysis.

It is not proposed to go further into this matter in the present volume, having said sufficient to leave little doubt that they are all consistent with one another.

The question as to whether the postulates are all independent is mainly a matter of logical nicety and is of comparatively little importance provided that the number of redundant postulates be not large.

In the course of development of the present work the writer succeeded in eliminating a considerable number of postulates which he had provisionally laid down: the redundancies being generally indicated by the possibility of proving some particular result from several sets of postulates.

One known redundancy has been permitted to remain: namely Post. II (a) and (b), which might have been deduced directly from Post. V and Post. VI (a) and (b).

By retaining Post. II, however, our first four postulates will be seen to hold for the set of instants of which any one individual is directly conscious, and the subject is thus better exhibited as an extension of the commonly accepted ideas of time.

A still further diminution of the number of postulates might have been made by combining Posts. VI and XI in the way mentioned on p. 42, but to have done so would have complicated still further the initial part of the subject, since Post. VI implies merely a two-dimensional conical order, while its combination with Post. XI makes the set of elements at least three-dimensional from the very beginning.

Apart from the above-mentioned, no further definite indications of redundancy have been observed, and, although some redundant postulates may still remain, it seems unlikely that there can be many.

This opinion is confirmed by a comparison with the number of fundamental assumptions given by various writers on the foundations of ordinary geometry.

We have now concluded the exposition of the argument by which we have been led to the view expressed in the introduction: that *spacial relations are to be regarded as the manifestation of the fact that the elements of time form a system in conical order: a conception which may be analysed in terms of the relations of* after *and* before.

This view would appear to have important bearings on general philosophy, but into these we do not purpose here to enter.

One point may however be mentioned:

The fundamental properties of time must, on any theory, be regarded as possessing a character which is not transitory, but in some sense persistent; since otherwise, statements about the past or future would be meaningless.

We here touch on the difficult problem as to the nature of "universals": a problem which has been much discussed by philosophers, but appears to be still far from a satisfactory solution.

Though space may be analysable in terms of time relations, yet these remain in their ultimate nature as mysterious as ever; and though events occur in time, yet any logical theory of time itself must always imply the Unchangeable.

APPENDIX

It is worthy of note that, just as by treating a system of co-directional inertia lines as points, we may represent ordinary Euclidean geometry in our time-space continuum: so by means of a system of inertia lines having a common element, we may represent the geometry of Lobatschewski in a very analogous manner.

We shall first prove a certain theorem with regard to three such inertia lines.

Let these be denoted by l_1, l_2 and l_3 and let them have the common element O.

Let A be any element of l_1 which is *after* O and let a separation line be taken through A normal to l_1 and lying in the inertia plane containing l_1 and l_2 and let it intersect l_2 in B'.

Similarly let a separation line be taken through A normal to l_1 and lying in the inertia plane containing l_1 and l_3 and let it intersect l_3 in C'.

Then, since OA is normal to both AB' and AC', it follows that AB' and AC' lie in a separation plane, so that $B'C'$ is a separation line and the relations of the sides and angles of the triangle whose corners are A, B', C' are the same as in ordinary Euclidean geometry, and we shall denote $\angle B'AC'$ by A.

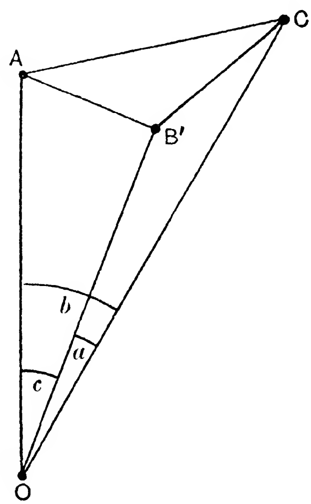


Fig. 57.

Also, since both B' and C' are neither *before* nor *after* A, while A is *after* O, it follows that both B' and C' are also *after* O, and accordingly, the inertia half-lines OA, OB' and OC' having the common end O must make proper hyperbolic angle-boundaries with one another.

We shall denote the proper hyperbolic angles $\angle C'OB'$, $\angle AOC'$ and $\angle B'OA$ by a, b and c respectively, and shall use the abbreviation *conj* for the word conjugate.

Then, as we have seen

$$(\text{conj } B'C')^2 = 2OB' \,.\, OC' \cosh a - OB'^2 - OC'^2.$$

Also
$$B'C'^2 = AB'^2 + AC'^2 - 2AB' \,.\, AC' \cos A;$$

so that

$$(\text{conj } B'C')^2 = (\text{conj } AB')^2 + (\text{conj } AC')^2 - 2\,(\text{conj } AB')(\text{conj } AC') \cos A.$$

Thus

$$2OB' \,.\, OC' \cosh a = OB'^2 + (\text{conj}\, AB')^2 + OC'^2 + (\text{conj}\, AC')^2 - 2\,(\text{conj}\, AB')\,(\text{conj}\, AC') \cos A,$$

or $$2OB' \,.\, OC' \cosh a = 2OA^2 - 2\,(\text{conj}\, AB')\,(\text{conj}\, AC') \cos A.$$

This may be written in the form

$$\cosh a = \frac{OA}{OC'} \cdot \frac{OA}{OB'} - \frac{(\text{conj}\, AC')}{OC'} \cdot \frac{(\text{conj}\, AB')}{OB'} \cos A,$$

or $$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos A \qquad \text{......(1)};$$

where $\cos A$ is equal to the cosine of the di-hedral angle which the two inertia planes containing l_1 make with one another.

By similar constructions taken with respect to l_2 and l_3, denoting the corresponding di-hedral angles by B and C respectively, we may deduce the equations

$$\cosh b = \cosh c \cosh a - \sinh c \sinh a \cos B \qquad \text{......(2)},$$

and $$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C \qquad \text{......(3)}.$$

These equations (1), (2) and (3) are the relations connecting the sides and angles of a triangle in the geometry of Lobatschewski.

Now let R be the set of elements which are *after* O exclusive of those which lie in the α sub-set of O, and consider the portions lying in the region R of all inertia lines, inertia planes and inertia threefolds which pass through the element O.

If then, for the purpose of this representation, we call such a portion of an inertia line a *point*; such a portion of an inertia plane a *line*, and such a portion of an inertia threefold a *plane*; we see that: any two points determine a line; while any three points which do not lie in one line determine a plane.

Also the sides and angles of any triangle satisfy the relations of the geometry of Lobatschewski.

It is to be observed that, since optical and separation lines which pass through O do not lie in the region R, two inertia planes passing through O and intersecting in an optical or separation line through O correspond to lines in a plane which have no point in common. In the case where the inertia planes intersect in an optical line they correspond to Lobatschewski parallels; since an optical half-line does not make a finite hyperbolic angle with any inertia half-line.

It is obvious that similar results hold if, instead of the region R, we take a region R' consisting of all elements which are *before* O exclusive of those which lie in the β sub-set of O.

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References in **dark** *type are to definitions*

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